

Solutions Manual-Theory of Vibration with Applications
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1-1

$$x = A \sin \omega t$$

$$A = 0.20 \text{ cm} \quad T = 0.15 \text{ s}$$

$$\omega = \frac{2\pi}{T} = 41.89 \text{ rad/s}$$

$$\dot{x} = \omega A \cos \omega t$$

$$\dot{x}_{\max} = \omega A = 8.38 \text{ cm/s}$$

$$\ddot{x} = -\omega^2 A \sin \omega t$$

$$\ddot{x}_{\max} = \omega^2 A = 350.9 \text{ cm/s}^2$$

1-2

$$\omega = 2\pi f = 2\pi \times 82 = 515.2 \text{ rad/s}$$

$$\omega^2 = 0.2655 \times 10^6$$

$$g = 980.4 \text{ cm/s}^2$$

$$x_{\max} = \ddot{x}_{\max} / \omega^2 = \frac{50 \times 980.4}{0.2655 \times 10^6} = 0.184 \text{ cm}$$

1-3

$$\omega = 2\pi f = 2\pi \times 10 = 62.83 \text{ rad/s}$$

$$T = \frac{1}{f} = 0.10 \text{ s}$$

$$\dot{x}_{\max} = \omega A = 4.57 \text{ m/s}$$

$$A = 0.07274 \text{ m}$$

$$= 7.274 \text{ cm}$$

$$\ddot{x}_{\max} = \omega^2 A = 287.1 \text{ m/s}^2$$

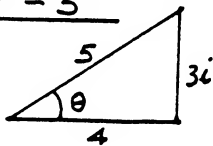
1-4

$$x = A (\sin \omega_1 t + \sin \omega_2 t) = 2A \cos \frac{1}{2}(\omega_1 - \omega_2)t \cdot \sin \frac{1}{2}(\omega_1 + \omega_2)t$$

$$\text{let } \omega_1 = \omega, \quad \omega_2 = \omega + \Delta\omega, \quad \omega_1 + \omega_2 = 2\omega + \Delta\omega \cong 2\omega$$

$$\therefore x \cong 2A \cos \frac{1}{2} \Delta\omega t \cdot \sin \omega t$$

1-5



$$z = 4 + 3i = 5(\cos \theta + i \sin \theta) = 5e^{i\theta}$$

$$\theta = \tan^{-1} \frac{3}{4} = 36^\circ 52' = 0.6435 \text{ rad}$$

1-6

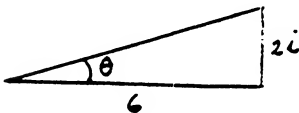
$$\begin{array}{r} 2 + 3i \\ 4 - i \\ \hline \text{Sum} = 6 + 2i \end{array}$$

$$z = A e^{i\theta} \quad A = \sqrt{6^2 + 2^2} = 6.325$$

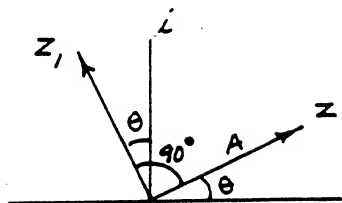
$$\theta = \tan^{-1} \frac{2}{6} = 18^\circ 26' = 0.3217 \text{ rad}$$

$$z = 6.325 e^{0.3217i}$$

$$= 6.325 \angle 18^\circ 26'$$



1-7

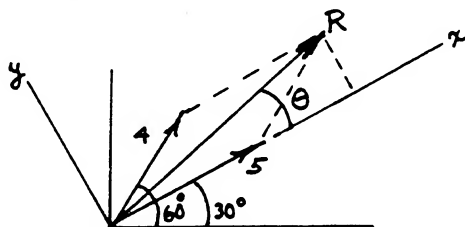


$$z = A (\cos \theta + i \sin \theta) = A e^{i\theta}$$

$$iz = A (i \cos \theta - \sin \theta) = z_1$$

$$= A [\cos(\theta + 90^\circ) + i \sin(\theta + 90^\circ)]$$

1-8



$$\frac{\pi}{6} = 30^\circ$$

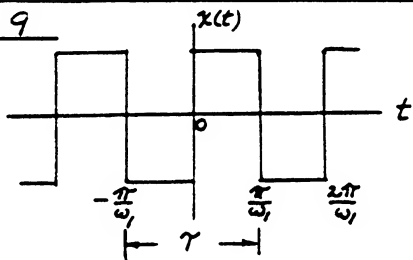
$$R_x = 5 + 4 \cos 30^\circ = 5 + 3.47 = 8.47$$

$$R_y = 4 \sin 30^\circ = 2.00$$

$$R = \sqrt{8.47^2 + 2.0^2} = 8.70$$

$$\theta = \tan^{-1} \frac{2}{8.70} = 12^\circ 57'$$

1-9



$x(t)$ is odd function $\therefore a_n = 0$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin n\omega_1 t dt$$

$$T = \frac{2\pi}{\omega_1}$$

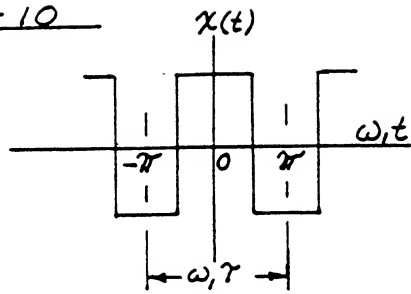
$$b_n = \frac{\omega_1}{\pi} \left[\int_{-\pi/\omega_1}^0 (-1) \sin n\omega_1 t dt + \int_0^{\pi/\omega_1} (+1) \sin n\omega_1 t dt \right]$$

$$= \frac{\omega_1}{\pi} \left[\left. \frac{\cos n\omega_1 t}{n\omega_1} \right|_{-\pi/\omega_1}^0 - \left. \frac{\cos n\omega_1 t}{n\omega_1} \right|_0^{\pi/\omega_1} \right] = \frac{2}{n\pi} (1 - \cos n\pi)$$

$$\therefore b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{for } n \text{ odd} \end{cases}$$

$$x(t) = \frac{4}{\pi} \left(\sin \omega_1 t + \frac{1}{3} \sin 3\omega_1 t + \frac{1}{5} \sin 5\omega_1 t + \dots \right)$$

1-10



$x(t)$ is even function

$$b_m = 0$$

$$a_m = \frac{2}{\gamma} \int_{-\gamma/2}^{\gamma/2} x(t) \cos m\omega_1 t dt$$

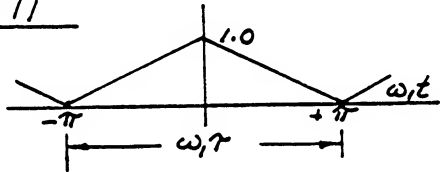
$$\gamma = \frac{2\pi}{\omega_1}, \quad \omega_m = n\omega_1$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{-\pi/2} (-1) \cos n\omega_1 t d(\omega_1 t) + \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} (+1) \cos n\omega_1 t d(\omega_1 t) + \frac{1}{\pi} \int_{\pi/2}^{\pi} (-1) \cos n\omega_1 t d(\omega_1 t)$$

$$= \frac{1}{n\pi} \left\{ \pm 4 \right\} \quad \begin{array}{l} + \text{ for } n = 1, 5, 9, \dots \\ - \text{ for } n = 3, 7, 11, \dots \end{array}$$

$$x(t) = \frac{4}{\pi} \left(\cos \omega_1 t - \frac{1}{3} \cos 3\omega_1 t + \frac{1}{5} \cos 5\omega_1 t - \dots \right)$$

1-11



$x(t)$ is even function $b_m = 0$

$$x(t) = \begin{cases} \frac{1}{\pi}(t+\pi) & -\pi \leq t \leq 0 \\ \frac{1}{\pi}(\pi-t) & 0 \leq t \leq \pi \end{cases}$$

$$\frac{1}{2} a_0 = \text{average value} = \frac{1}{2}$$

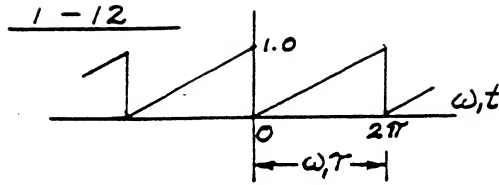
$$a_m = \frac{1}{\pi} \int_{-\pi}^0 \frac{1}{\pi}(t+\pi) \cos n\omega_1 t d(\omega_1 t) + \frac{1}{\pi} \int_0^{\pi} \frac{1}{\pi}(\pi-t) \cos n\omega_1 t d(\omega_1 t)$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{\pi}(\pi-t) \cos n\omega_1 t d(\omega_1 t) \quad \text{because of symmetry}$$

$$= \frac{2}{\pi} \left. \frac{\sin n\omega_1 t}{n} \right|_0^{\omega_1 t = \pi} - \frac{2}{\pi^2} \left(\frac{\cos n\omega_1 t}{n^2} + \omega_1 t \frac{\sin n\omega_1 t}{n} \right) \Big|_0^{\omega_1 t = \pi}$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n^2 \pi^2} & \text{for } n \text{ odd} \end{cases}$$

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2} \left(\cos \omega_1 t + \frac{1}{3^2} \cos 3\omega_1 t + \frac{1}{5^2} \cos 5\omega_1 t + \dots \right)$$



$$x = \frac{\omega_1 t}{2\pi} \quad 0 \leq \omega_1 t \leq 2\pi$$

$$C_n = \frac{\omega_1}{2\pi} \int_0^{2\pi} \frac{\omega_1 t}{2\pi} e^{-in\omega_1 t} dt$$

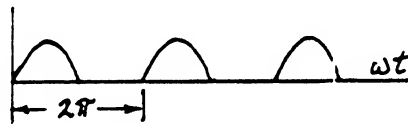
$$C_n = \frac{1}{(2\pi)^2} \int_0^{2\pi} \omega_1 t e^{-in\omega_1 t} d(\omega_1 t) = \frac{1}{(2\pi)^2} \left[\frac{e^{-in\theta}}{(-in)^2} (-in\theta - 1) \right] \text{ where } \theta = \omega_1 t$$

$$= \frac{1}{(2\pi)^2 n^2} \left[-1 + (1 + i2\pi n) e^{-i2\pi n} \right] = \frac{i}{2\pi} \frac{1}{n}$$

$$x(t) = C_0 + \frac{i}{2\pi} \left[(e^{i\omega_1 t} - e^{-i\omega_1 t}) + \frac{1}{2} (e^{i2\omega_1 t} - e^{-i2\omega_1 t}) + \frac{1}{3} (e^{i3\omega_1 t} - e^{-i3\omega_1 t}) + \dots \right]$$

$$= \frac{1}{2} - \frac{1}{\pi} \left[\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \frac{1}{3} \sin 3\omega_1 t + \dots \right]$$

1-13



$$\overline{x^2} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} A^2 \sin^2 \omega t dt = \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_0^{T/2} \frac{1}{2} (1 - \cos 2\omega t) dt$$

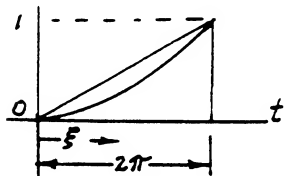
$$= \lim_{T \rightarrow \infty} \frac{A^2}{2} \left(\frac{t}{2} - \frac{\sin 2\omega t}{4\omega} \right) \Big|_0^{T/2} = \lim_{T \rightarrow \infty} \left(\frac{A^2}{4} - \frac{\sin \omega T}{\omega T} \right) = \frac{A^2}{4}$$

$$\therefore x_{RMS} = \sqrt{\overline{x^2}} = \frac{A}{2}$$

1-14

$$x(t) = \frac{\xi}{2\pi}$$

$$x^2(t) = \frac{1}{4\pi^2} \xi^2 \quad 0 \leq \xi \leq 2\pi$$



$$\overline{x^2} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{4\pi^2} \xi^2 \right) d\xi = \frac{1}{2\pi} \left(\frac{1}{4\pi^2} \frac{\xi^3}{3} \right) \Big|_0^{2\pi}$$

$$= \frac{1}{2\pi} \left(\frac{2\pi}{3} \right) = \frac{1}{3}$$

1-14 cont.

F.S. of saw tooth wave =

$$x(t) = \frac{1}{2} - \frac{1}{\pi} \left(\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \frac{1}{3} \sin 3\omega_1 t + \dots \right)$$

$$x^2(t) = \frac{1}{4} - \frac{1}{\pi} \left(\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \dots \right)$$

$$+ \frac{1}{\pi^2} \left(\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \dots \right)^2$$

$$= \frac{1}{4} - \frac{1}{\pi} \left(\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \dots \right)$$

$$+ \frac{1}{\pi^2} \left(\sin^2 \omega_1 t + \sin^2 2\omega_1 t + \dots \right) + \text{cross products which will integrate to zero}$$

$$\begin{aligned} \frac{1}{T} \int_0^T x^2(t) dt &= \frac{1}{4} + \frac{1}{\pi \omega_1 T} \cos \omega_1 t \Big|_0^T + \frac{1}{4\pi \omega_1 T} \cos 2\omega_1 t \Big|_0^T + \frac{1}{9\pi \omega_1 T} \cos 3\omega_1 t \Big|_0^T \\ &+ \frac{1}{\pi^2 T} \frac{1}{2} \left(t - \frac{\sin 2\omega_1 t}{2\omega_1} \right) \Big|_0^T + \frac{1}{4\pi^2 T} \frac{1}{2} \left(t - \frac{\sin 4\omega_1 t}{4\omega_1} \right) \Big|_0^T + \dots \end{aligned}$$

Let $\omega_1 T = 2\pi k$ where k is an integer $\rightarrow \infty$

$$\overline{x^2} = \lim_{k \rightarrow \infty} \frac{1}{2\pi k} \int_0^{2\pi k} x^2(t) dt = \frac{1}{4} + \frac{1}{2\pi^2} + \frac{1}{2} \frac{1}{(2\pi)^2} + \frac{1}{2} \frac{1}{(3\pi)^2} + \dots$$

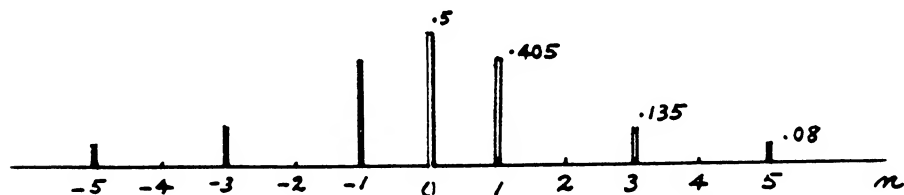
$$= \frac{1}{4} + \frac{1}{2\pi^2} \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \right) = \frac{1}{3}$$

1-15

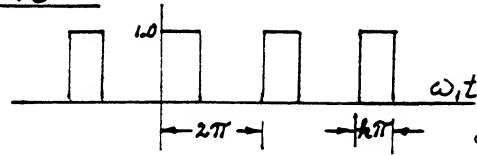
$$x(t) = \frac{1}{2} + \frac{4}{\pi^2} \left(\cos \omega_1 t + \frac{1}{3} \cos 3\omega_1 t + \frac{1}{5} \cos 5\omega_1 t + \dots \right)$$

Fourier Spectrum = plot of coefficients. For this case $b_n = 0$

$$C_n = \sqrt{a_n^2 + b_n^2} = a_n \quad C_0 = \frac{a_0}{2}$$



1-16



$$k = \frac{2}{3}$$

$$C_0 = \frac{A_0}{2} = \text{average value} = \frac{k\pi}{2\pi} = \frac{k}{2} = \frac{1}{3}$$

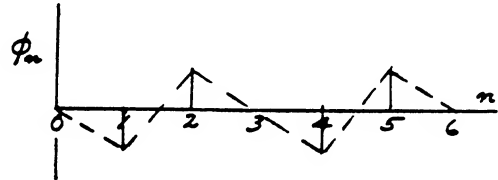
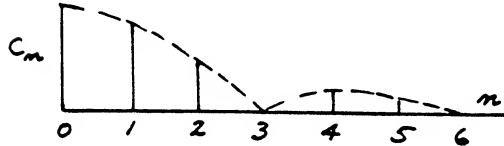
$$a_m = \frac{1}{m\pi} \sin m k \pi = \frac{1}{m\pi} \sin \frac{m}{3} 2\pi$$

$$b_m = \frac{1}{m\pi} (1 - \cos m k \pi) = \frac{1}{m\pi} (1 - \cos \frac{m}{3} 2\pi)$$

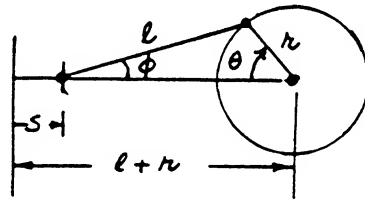
$$2C_m = \sqrt{a_m^2 + b_m^2} = \frac{\sqrt{2}}{m\pi} \sqrt{(1 - \cos \frac{m}{3} 2\pi)} \quad \phi_m = \tan^{-1} \frac{1 - \cos \frac{m}{3} 2\pi}{\sin \frac{m}{3} 2\pi}$$

n	C_m	ϕ_m
1	.2758	-60°
2	.1379	60°
3	0	0

n	C_m	ϕ_m
4	.0689	-60°
5	.0552	60°
6	0	0



1-17



$$l + r - S = r \cos \theta + l \cos \phi$$

$$l \sin \phi = r \sin \theta$$

$$\therefore \cos \phi = \left[1 - \left(\frac{r}{l} \right)^2 \sin^2 \theta \right]^{1/2}$$

$$= 1 - \frac{1}{2} \left(\frac{r}{l} \right)^2 \sin^2 \theta - \frac{1}{8} \left(\frac{r}{l} \right)^4 \sin^4 \theta \dots$$

$$S = r \left[1 - \cos \theta + \frac{1}{2} \left(\frac{r}{l} \right) \sin^2 \theta + \frac{1}{8} \left(\frac{r}{l} \right)^3 \sin^4 \theta + \dots \right]$$

$$\text{using } \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta), \quad \sin^4 \theta = \frac{1}{4} \left(\frac{3}{2} - 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right)$$

$$S = r \left[1 - \cos \theta + \frac{1}{2} \left(\frac{r}{l} \right) \frac{1}{2} (1 - \cos 2\theta) + \dots \right]$$

$$= r \left[1 + \frac{1}{4} \left(\frac{r}{l} \right) - \cos \theta - \frac{1}{4} \left(\frac{r}{l} \right) \cos 2\theta + \dots \right] \quad \text{which retains only } \left(\frac{r}{l} \right) \text{ to first power}$$

$$\text{Ratio of 2}^{\text{nd}} \text{ harmonic} / 1^{\text{st}} \text{ harmonic} = \frac{1}{4} \left(\frac{r}{l} \right) = \frac{1}{12}$$

$$\frac{1-18}{\overline{x^2}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt = \frac{A^2 k \gamma}{\gamma} = k A^2 = 0.10 A^2$$

$$r.m.s = \sqrt{\overline{x^2}} = 0.3162 A$$

$$\frac{1-19}{x = (1 - \frac{t}{\pi}) \quad 0 \leq t \leq \pi}$$

$$x^2 = 1 - \frac{2t}{\pi} + \frac{t^2}{\pi^2}$$

$$\overline{x^2} = \frac{1}{\pi} \int_0^{\pi} (1 - \frac{2t}{\pi} + \frac{t^2}{\pi^2}) dt = \frac{1}{3}$$

$$\frac{1-20}{}$$

$$Db = 20 \log_{10} \left(\frac{x_1}{x_2} \right) = 0.50$$

$$\log_{10} \left(\frac{x_1}{x_2} \right) = \frac{0.50}{20} = 0.0250$$

$$\left(\frac{x_1}{x_2} \right) = 10^{0.0250} = 1.0593$$

$$x_1 = 1.0593 x_2 = 1.0593 \times 2.5 \text{ mm} = 2.6481$$

$$\text{Error} = 0.0593 \times 2.5 \text{ mm} = \pm 0.148 \text{ mm}$$

$$\frac{1-21}{}$$

$$Db = 20 \log_{10} (10) = 20$$

$$Db = 20 \log_{10} (50) = 33.98$$

$$Db = 20 \log_{10} (100) = 40.0$$

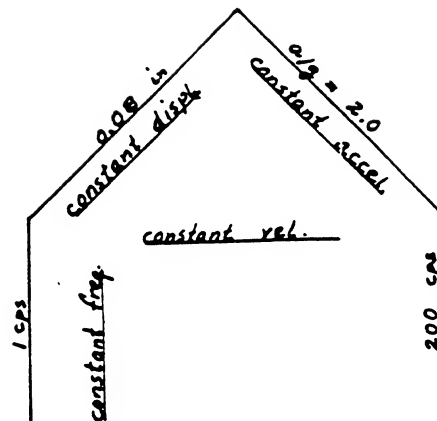
$$\frac{1-22}{}$$

$$Db = 20 \log_{10} \left(\frac{x_p}{x_{1000}} \right) = 32.$$

$$\log_{10} \left(\frac{x_p}{x_{1000}} \right) = \frac{32}{20} = 1.60$$

$$\frac{x_p}{x_{1000}} = 10^{1.60} = 39.8$$

1-23



1-24

$$\text{Mean Value} = 1 * P(1) + (-1) * P(-1) = 0$$

$$\text{Mean Square Value} = (1)^2 * P(1) + (-1)^2 * P(-1) = 1$$

1-25

$$f(t) = \frac{1}{2} (f(t) - f(-t)) + \frac{1}{2} (f(t) + f(-t))$$

$$= O(t) + E(t)$$

$$O(-t) = \frac{1}{2} (f(-t) - f(t)) = -O(t) \quad (\text{odd function})$$

$$E(-t) = \frac{1}{2} (f(-t) + f(t)) = E(t) \quad (\text{even function})$$

2-1 From Eq. 2.2-9

$$f = \frac{15.76}{\sqrt{\Delta \text{ mm}}} = \frac{15.76}{\sqrt{7.87}} = 5.62 \text{ Hz}$$

2-2



$$f_1 = \sqrt{\frac{k_1}{m}}$$

$$f_2 = \sqrt{\frac{k_1 + k_2}{m}}$$

$$\frac{1}{2} \sqrt{\frac{k_1}{m}} = \sqrt{\frac{k_1 + k_2}{m}}$$

$$\frac{1}{4} k_1 = k_1 + k_2 \quad \therefore k_2 = -\frac{3}{4} k_1$$

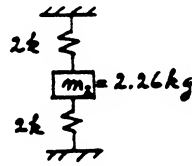
2-3



$$\gamma_1 = 0.45$$

$$k = \left(\frac{2\pi}{\gamma_1} \right)^2 m_1 = \left(\frac{2\pi}{0.45} \right)^2 4.53$$

$$= 883.5 \text{ N/m}$$



$$\gamma_2 = 2\pi \sqrt{\frac{m_2}{4k}} = 2\pi \sqrt{\frac{2.26}{4 \times 883.5}}$$

$$= 0.159$$

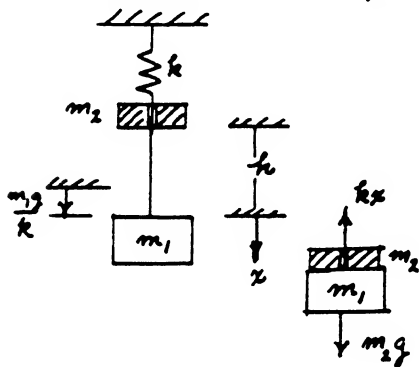
2-4

$$\frac{k}{m} = (2\pi f)^2 = \left(2\pi \frac{94}{60}\right)^2 \quad \frac{k}{m+0.453} = \left(2\pi \frac{76.7}{60}\right)^2$$

$$\frac{m+0.453}{m} = \left(\frac{94}{76.7}\right)^2 \quad \therefore m = 0.9028 \text{ kg}$$

$$k = 87.48 \text{ N/m}$$

2-5



x measured from static equilibrium position of m_1 , k

Eq. of motion after impact

$$(m_1+m_2)\ddot{x} = -kx + m_2g$$

Gen solution:

$$x(t) = \frac{m_2g}{k} + A\sin\omega t + B\cos\omega t$$

Initial conditions

$$x(0) = 0 = \frac{m_2g}{k} + B \quad \therefore B = -\frac{m_2g}{k}$$

$$\dot{x}(0) = \frac{m_2\sqrt{2gh}}{m_1+m_2} = \omega A \quad \therefore A = \frac{m_2\sqrt{2gh}}{(m_1+m_2)\omega}$$

$$\omega = \sqrt{\frac{k}{m_1+m_2}}$$

$$\therefore x(t) = \frac{m_2g}{k} + \frac{m_2\sqrt{2gh}}{m_1+m_2} \sqrt{\frac{m_1+m_2}{k}} \sin\omega t - \frac{m_2g}{k} \cos\omega t$$

$$= \frac{m_2g}{k}(1 - \cos\omega t) + \frac{m_2\sqrt{2gh}}{\sqrt{k(m_1+m_2)}} \sin\omega t$$

2-6

$$\omega_m^2 = \frac{k}{m} = 4.0$$

$$\omega_m = 2.0$$

$$x = x_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t = 2 \cos 2t - \frac{8}{2} \sin 2t$$

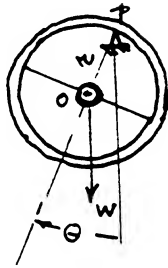
$$\dot{x} = -4 \sin 2t - 8 \cos 2t = 0 \quad \therefore \tan 2t_p = -2$$

$$\therefore 2t_p = 116.57^\circ \quad \sin 116.57^\circ = 0.8944 \quad \cos 116.57^\circ = -0.4472$$

$$x_{\max} = 2(-0.4472) - 4(0.8944) = -4.472 \text{ cm}$$

$$\ddot{x}_{\max} = \omega^2 x_{\max} = 4(\pm 4.472) = \pm 17.89 \text{ cm/s}^2$$

2-7



$$J_p \ddot{\theta} = -W r \sin \theta \quad \ddot{\theta} = -\omega^2 \theta$$

$$J_p = \frac{W r}{\omega^2} = \frac{70 \times 6}{\left(\frac{2\pi}{1.22}\right)^2} = 15.83$$

$$J_o = J_p - \frac{W}{g} r^2 = 15.83 - \frac{70}{386} \times 6^2 = 9.30 \text{ lb in sec}^2$$

2-8

$$\omega = 2\pi \frac{53}{60} = 5.55 \text{ rad/s}$$

$$J_p = \frac{W r}{\omega^2} = \frac{21.35 \times 0.254}{5.55^2} = 0.1761$$

$$J_{cg} = J_o - \frac{W r^2}{g} = 0.1761 - \frac{21.35 \times 0.254^2}{9.81} = 0.0356 \text{ kg m}^2$$

2-9

$$r\theta = l\alpha$$

$$\alpha = \frac{r\theta}{l}$$



vertical displ =

$$l(1 - \cos \alpha) = l\left[1 - \left(1 - \frac{1}{2}\alpha^2 + \dots\right)\right] = \frac{l\alpha^2}{2} = \frac{l}{2}\left(\frac{r\theta}{l}\right)^2$$

Work done = change in KE

$$W \frac{l}{2} \frac{r^2}{l^2} \theta_{\max}^2 = \frac{1}{2} J \dot{\theta}_{\max}^2 = \frac{1}{2} J \omega^2 \theta_{\max}^2$$

$$J = \frac{W}{g} k^2 \quad k = \text{rad. of gyr.}$$

$$W \frac{r^2}{l} = \frac{W}{g} k^2 \omega^2$$

$$k^2 = \frac{r}{\omega} \sqrt{\frac{g}{l}} = \frac{0.254 \times 2.17}{2\pi} \sqrt{\frac{9.81}{1.829}} = 0.2032$$

$$k = 0.4507 \text{ m}$$

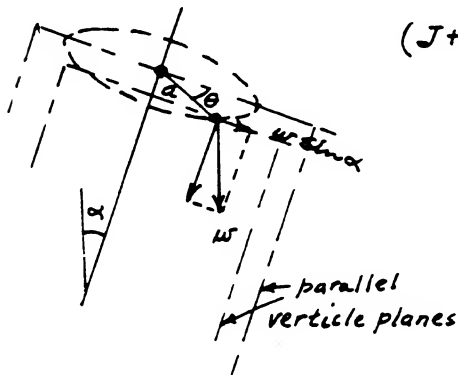
2-10

Moment about shaft = $(a \sin \theta) w \sin \alpha$

$$\left(J + \frac{w}{g} a^2\right) \ddot{\theta} = -(a \sin \theta) w \sin \alpha$$

$$\approx -(a w \sin \alpha) \theta$$

$$\therefore f_n = \frac{1}{2\pi} \sqrt{\frac{w a \sin \alpha}{J + \frac{w}{g} a^2}}$$



2-11

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 \quad k\theta = \dot{x}$$

$$= \frac{1}{2} \left(m + \frac{J_0}{R^2} \right) \dot{x}^2 \quad \dot{x} = \omega x$$

$$U = \frac{1}{2} k x^2 \quad \therefore \omega = \sqrt{\frac{k}{m + J_0/R^2}}$$

2-12

$$\gamma = 2\pi \sqrt{\frac{L}{g}} \quad L = g \left(\frac{\gamma}{2\pi} \right)^2 = 9.81 \left(\frac{2}{2\pi} \right)^2 = 0.994 \text{ m}$$

$$v_{\max} = L(\omega \theta_0) = \frac{.003175}{.01} \text{ m/s} \quad \theta_0 = \frac{.3175}{.994 \pi} = .1017 \text{ rad} = 5.826^\circ$$

2-13

$$\text{water weighs } 9802 \text{ N/m}^3 \quad \therefore \rho = 1.2 \times 9802 = 11762 \text{ N/m}^3$$

$$\text{buoyant force} = \pi R^2 x \cdot \rho = m \ddot{x} = \omega^2 x$$

$$\frac{1}{\omega} = \frac{\gamma}{2\pi} = \sqrt{\frac{m}{\pi R^2 \rho}} \quad m = .0372 \text{ kg}$$

$$R = .0032 \text{ m}$$

$$\gamma = 2\pi \sqrt{\frac{.0372}{\pi \times .0032^2 \times 11762}} = 1.97 \text{ s}$$

2-14

moment about geom. center

$$-W \delta \theta = J_0 \ddot{\theta} = -\omega^2 J_0 \theta$$

$$J_0 = \frac{8W}{\omega^2} = \frac{8W (1.3)^2}{(2\pi)^2} = 0.3428 W$$

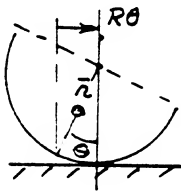
2-15

$$-W h \theta = J \ddot{\theta} = -\omega^2 J \theta$$

$$\frac{1}{\omega} = \frac{\gamma}{2\pi} = \sqrt{\frac{J}{Wh}} \quad \gamma = 2\pi \sqrt{\frac{J}{Wh}}$$

2-16Displ. of cg = $(R - \bar{r})\theta$

$$T_{\max} = U_{\max}$$



$$T_{\max} = \frac{1}{2} m (R - \bar{r})^2 \dot{\theta}_{\max}^2 + \frac{1}{2} J_{cg} \dot{\theta}_{\max}^2$$

$$= \frac{1}{2} m [(R - \bar{r})^2 + (R^2 - \bar{r}^2)] \omega^2 \theta_{\max}^2$$

$$U_{\max} = mg \bar{r} (1 - \cos \theta_{\max}) \approx mg \bar{r} \frac{\theta_{\max}^2}{2}$$

2-16 cont

$$T_{max} = U_{max}$$

$$\omega^2 = \frac{\bar{h}g}{(R-\bar{h})^2 + (R^2 - \bar{h}^2)} = \frac{\bar{h}g}{2R(R-\bar{h})}$$

$$\gamma = 2\pi \sqrt{\frac{2R(R-\bar{h})}{\bar{h}g}} \quad \text{but } \bar{h} = \frac{2R}{\pi} \quad \therefore \gamma = 2\pi \sqrt{\frac{R(\pi-2)}{g}}$$

2-17

$$U = mgh(1 - \cos \phi) \approx mgh \frac{1}{2} \phi^2 \quad h\phi = \frac{a}{2} \theta$$

$$= mgh \frac{1}{2} \left(\frac{a\theta}{2h} \right)^2 = mgh \frac{a^2}{8} \frac{\theta^2}{h}$$

$$T = \frac{1}{2} \left(m \frac{L^2}{12} \right) \dot{\theta}^2 = \frac{1}{2} \left(m \frac{L^2}{12} \right) \omega^2 \theta^2$$

$$T_{max} = U_{max} \quad \therefore \gamma = 2\pi \frac{L}{a} \sqrt{\frac{h}{3g}}$$

2-18

$$\gamma_1 = 2\pi \sqrt{\frac{h}{g}}$$

for γ_2

$$T = \frac{1}{2} m \kappa^2 \omega^2 \theta^2$$

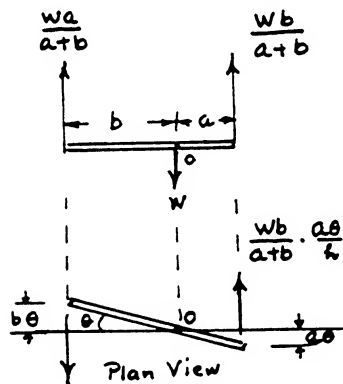
$$U = mgh \frac{1}{8} \frac{\theta^2}{h}$$

$$\therefore \gamma_2 = 2\pi \sqrt{\frac{4h\kappa^2}{gL^2}}$$

$$\kappa = \frac{\gamma_2 L}{2\pi} \sqrt{\frac{g}{4h}} = \frac{\gamma_2}{\gamma_1} \left(\frac{L}{2} \right)$$

2-19

$$\sum M_o = J_o \ddot{\theta} = - \frac{Wba}{a+b} \frac{\theta}{h} \cdot a - \frac{Wab}{a+b} \frac{\theta}{h} \cdot b$$



$$\frac{W}{g} \kappa^2 \ddot{\theta} + \frac{Wab}{h} \theta = 0$$

$$\ddot{\theta} + \left(\frac{gab}{\kappa^2 h} \right) \theta = 0$$

$$\therefore f = \frac{1}{2\pi} \sqrt{\frac{gab}{\kappa^2 h}}$$

$$\frac{Wa}{a+b} \cdot \frac{b\theta}{h}$$

$$\therefore \sum F_{horiz} = 0$$

2-20

$$J_o \text{ of wheel about torsion bar} = J_{cm} + m(24'')^2 \\ = m(k^2 + 24^2) = m(9^2 + 24^2) = 657 m$$

$$\text{Stiffness of torsion bar } K = \frac{G I_p}{l}$$

$$I_p = \frac{\pi D^4}{32} = \frac{\pi (1.50)^4}{32} = 0.497 \text{ in}^4 = \text{polar mom. inertia of torsion bar}$$

$$G = 11.2 \times 10^6 \text{ lb/in}^2 = \text{shear modulus of steel}$$

$$K = \frac{(11.2 \times 10^6) \times (0.497)}{50} = 0.1113 \times 10^6 \text{ lb.in./rad.}$$

$$J_o \ddot{\theta} + K \theta = 0 \quad f = \frac{1}{2\pi} \sqrt{\frac{K}{J_o}} = \frac{1}{2\pi} \sqrt{\frac{0.1113 \times 10^6 \times 386}{38 \times 657}} = 6.60 \text{ cps}$$

locked wheel.

$$\text{With wheel free } J_o = m(24)^2 = 576 m$$

$$f = \frac{1}{2\pi} \sqrt{\frac{0.1113 \times 10^6 \times 386}{38 \times 576}} = 7.05 \text{ c.p.s.}$$

2-21

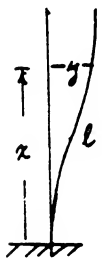
$$\frac{l\rho}{3} \ddot{x} = -2x\rho \quad \ddot{x} + \frac{2g}{l} x = 0$$

$$\omega^2 = \frac{2g}{l} \quad \gamma = 2\pi \sqrt{\frac{l}{2g}}$$

2-22

$$k = 2 \left(\frac{12EI}{l^3} \right) \quad \gamma = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{ml^3}{24EI}}$$

2-23



$$y = \frac{1}{2} y_{max} (1 - \cos \frac{\pi x}{l}) \sin \omega t$$

$$\dot{y} = \frac{1}{2} \omega y_{max} (1 - \cos \frac{\pi x}{l}) \cos \omega t$$

$$T = \frac{1}{2} \int_0^l m(x) \dot{y}^2 dx = \frac{1}{2} \frac{m}{4} y_{max}^2 \int_0^l (1 - \cos \frac{\pi x}{l})^2 \omega^2 dx \cos^2 \omega t$$

$$= \frac{1}{2} \cdot \frac{m}{4} y_{max}^2 \omega^2 \cos^2 \omega t \int_0^l (1 - 2 \cos \frac{\pi x}{l} + \cos^2 \frac{\pi x}{l}) dx$$

$$= \quad \quad \quad \left[x - \frac{2l}{\pi} \sin \frac{\pi x}{l} + \frac{x}{2} + \frac{1}{2} \cdot \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right]_0^l$$

2-23 Cont

$$T = \quad " \quad \left[\frac{3}{2} l - 0 + 0 \right] \\ = \frac{1}{2} \left(\frac{m}{4} \cdot \frac{3l}{2} \right) \omega^2 y_{\max}^2 \cos^2 \omega t$$

$$\therefore m_{\text{eff}} = \left(\frac{3}{8} ml \right) \text{ for each column, where} \\ ml = \text{total mass of each column}$$

2-24

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \left(\frac{\dot{x}}{b} \right)^2 \quad \text{where } J = \text{moment of} \\ \text{inertia of linkage about} \\ \text{pivot.}$$

$$x = \frac{b}{a} x_m$$

$$T = \frac{1}{2} m \left(\frac{b}{a} \right)^2 \dot{x}_m^2 + \frac{1}{2} J \left(\frac{b}{a} \right)^2 \frac{1}{b^2} \dot{x}_m^2$$

$$= \frac{1}{2} \left[m \left(\frac{b}{a} \right)^2 + \frac{J}{a^2} \right] \dot{x}_m^2 \quad \therefore m_{\text{eff}} = \left[m \left(\frac{b}{a} \right)^2 + \frac{J}{a^2} \right]$$

2-25

$$T = \frac{1}{2} [J_0 \dot{\theta}^2 + m_1 (b \dot{\theta})^2] = \frac{1}{2} [J_0 + m_1 b^2] \dot{\theta}^2$$

$$\dot{\theta} = \dot{x}/b \quad T = \frac{1}{2} [J_0/b^2 + m_1] \dot{x}^2 \quad m_{\text{eff}} = J_0/b^2 + m_1$$

2-26 The kinetic energy is

$$T = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m_r (b \dot{\theta})^2 + \frac{1}{2} \left(\frac{m_s}{3} \right) (b \dot{\theta})^2 = \frac{1}{2} (J + m_r b^2 + \frac{1}{3} m_s b^2) \dot{\theta}^2$$

With the velocity at A equal to $\dot{x} = a \dot{\theta}$, T becomes

$$T = \frac{1}{2} \left(\frac{J + m_r b^2 + \frac{1}{3} m_s b^2}{a^2} \right) \dot{x}^2 \quad \text{and the effective mass at A is}$$

$$m_A = \left(\frac{J + m_r b^2 + \frac{1}{3} m_s b^2}{a^2} \right)$$

2-27 $y = \frac{1}{2} y_0 \left[3 \left(\frac{x}{l} \right)^2 - \left(\frac{x}{l} \right)^3 \right]$

$$T = \frac{1}{2} m \int_0^l \dot{y}^2 dx = \frac{1}{2} \frac{m}{4} \dot{y}_0^2 \int_0^l \left[9 \left(\frac{x}{l} \right)^4 - 6 \left(\frac{x}{l} \right)^5 + \left(\frac{x}{l} \right)^6 \right] dx$$

$$= \frac{1}{2} m \dot{y}_0^2 \frac{l}{4} \left[\frac{9}{5} - 1 + \frac{1}{7} \right] = \frac{1}{2} \left(\frac{33}{140} ml \right) \dot{y}_0^2$$

2-28 $y(x) = \frac{wl}{24} \frac{l^3}{EI} \left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3 \right]$ x measured from free end.

$$y_0 = y_{\max} = \frac{wl}{8} \frac{l^3}{EI} \quad \therefore y(x) = \frac{1}{3} y_{\max} \left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3 \right]$$

$$T = \frac{1}{2} m \int_0^l \dot{y}^2(x) dx = \frac{1}{2} m \frac{1}{9} \dot{y}_{\max}^2 \int_0^l \left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3 \right]^2 dx$$

$$\begin{aligned} \int_0^l \left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right) + 3 \right]^2 dx &= \int_0^l \left[\left(\frac{x}{l}\right)^8 - 8\left(\frac{x}{l}\right)^5 + 6\left(\frac{x}{l}\right)^4 + 16\left(\frac{x}{l}\right)^2 - 24\left(\frac{x}{l}\right) + 9 \right] dx \\ &= l \left[\frac{1}{9}\left(\frac{x}{l}\right)^9 - \frac{8}{6}\left(\frac{x}{l}\right)^6 + \frac{6}{5}\left(\frac{x}{l}\right)^5 + \frac{16}{3}\left(\frac{x}{l}\right)^3 - \frac{24}{2}\left(\frac{x}{l}\right)^2 + 9\left(\frac{x}{l}\right) \right]_0^l = \frac{624}{270} l \end{aligned}$$

$$\therefore T = \frac{1}{2} m \frac{1}{9} \dot{y}_{\max}^2 \left(\frac{624}{270} \right) l = \frac{1}{2} (.2568 ml) \dot{y}_{\max}^2$$

Compare with Prob. 2-27

$$T = \frac{1}{2} \left(\frac{33}{140} \right) ml \dot{y}_0^2 = \frac{1}{2} (.2357 ml) \dot{y}_{\max}^2$$

Effective mass for the two problems are nearly equal

2-29 Let θ_0 = rotation of J , T_0 = total torque

$$T_L = \text{torque to left of } J$$

$$T_R = \text{ " " right " " }$$

$$\left(\frac{1}{K_1} + \frac{1}{K_2} \right) T_L = \theta_0 \quad \left(\frac{1}{K_2} \right) T_R = \theta_0$$

2-29 Cont.

$$T_o = T_L + T_R = \left[\frac{1}{\left(\frac{1}{K_1} + \frac{1}{K_2}\right)} + \frac{1}{\left(\frac{1}{K_2}\right)} \right] \theta_o = K \theta_o$$

$$\therefore K = \left(\frac{K_1 K_2}{K_1 + K_2} + K_2 \right) \quad \omega_n = \sqrt{\frac{K}{J}} = \frac{2\pi}{T}$$

2-30

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} J_2 \left(\frac{\dot{x}}{R_2} \right)^2 + \frac{1}{2} J_3 \left(\frac{R_2}{R_1} \frac{\dot{x}}{R_2} \right)^2$$

$$= \frac{1}{2} \left[m_1 + J_2/R_2^2 + J_3/R_1^2 \right] \dot{x}^2 = \frac{1}{2} m_{\text{eff}} \dot{x}^2$$

$$U = \frac{1}{2} k_1 x^2 + \frac{1}{2} K_3 \left(\frac{x}{R_1} \right)^2 = \frac{1}{2} \left[k_1 + K_3/R_1^2 \right] x^2 = \frac{1}{2} k_{\text{eff}} x^2$$

2-31

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \left(\frac{R_1}{R_2} \dot{\theta}_1 \right)^2$$

$$= \frac{1}{2} \left[J_1 + J_2 \left(\frac{R_1}{R_2} \right)^2 \right] \dot{\theta}_1^2 = \frac{1}{2} J_{\text{eff}} \dot{\theta}_1^2$$

2-32

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} (m_0 + m_2) \dot{x}^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$

$$\dot{\theta}_1 = \dot{x}/R_1, \quad \dot{\theta}_2 = \dot{x}/R_2$$

$$T = \frac{1}{2} \left[J_1/R_1^2 + (m_0 + m_2) + J_2/R_2^2 \right] \dot{x}^2$$

$$U = \frac{1}{2} k_1 (R_1 \theta_1)^2 + \frac{1}{2} k_2 (R_2 \theta_2)^2$$

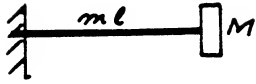
$$= \frac{1}{2} k_1 \left(\frac{R_1}{R_1} x \right)^2 + \frac{1}{2} k_2 \left(R_2 \frac{x}{R_2} \right)^2$$

$$= \frac{1}{2} \left[k_1 \left(\frac{R_1}{R_1} \right)^2 + k_2 \right] x^2$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k_1 \left(\frac{R_1}{R_1} \right)^2 + k_2}{J_1/R_1^2 + (m_0 + m_2) + J_2/R_2^2}}$$

2-33

$$m_{eff} = M + \frac{33}{140} ml \quad (\text{See Prob. 2-27})$$



$$\text{beam vol} = (.1016 \times .635 \times 8.89) = .5735 \text{ cm}^3$$

$$\text{wt. of steel} = 0.07655 \text{ N/cm}^3$$

$$\text{wt. of beam} = .5735 \times .07655 = .04390 \text{ N}$$

$$\text{mass of beam} = \frac{.04390}{9.81} = .00475 \text{ kg} = ml$$

$$\frac{33}{140} ml = .001055 \quad \text{beam stiffness} = \frac{3EI}{l^3} = k$$

$$E = 200 \times 10^9 \text{ N/m}^2 \quad I = \frac{bl^3}{12} = \frac{.635 \times .1016^3}{12} = .0000553 \times 10^{-8} \text{ m}^4$$

$$k = \frac{3 \times 200 \times 10^9 \times 553 \times 10^{-15}}{(.0889)^3} = 473.96 \text{ N/m}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m_{eff}}} \quad m_{eff} = M + .001055 = \frac{3EI}{l^3 4\pi^2 f^2} = \frac{473.96}{4\pi^2 \times 400}$$

$$\therefore M = 0.0289 \text{ kg}$$

2-34

$$C_c = 2 \sqrt{mk} = 2 \sqrt{.907 \times 7. \times 10^2} = 50.4 \frac{\text{Ns}}{\text{m}}$$

2-35

$$F_d = C v \quad C = \frac{F_d}{v} = \frac{0.50}{1.20} = 0.417 \frac{\text{lb sec}}{\text{in}}$$

$$C = 0.417 \frac{\text{lb. s}}{\text{in}} \times 4.448 \frac{\text{N}}{\text{lb}} \times \frac{1}{2.54} \frac{\text{in}}{\text{cm}} = 0.7303 \frac{\text{Ns}}{\text{cm}} = 73.03 \frac{\text{Ns}}{\text{m}}$$

$$\zeta = \frac{C}{C_c} = \frac{73.03}{50.4} = 1.45$$

2-36

$$(a) \quad \zeta = 2. \quad \text{Eq 2.6-20, 2.6-21} \quad A = -B = \frac{v_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

$$\frac{x \omega_n}{v_0} = \frac{1}{3.464} (e^{-0.268 \omega_n t} - e^{-3.732 \omega_n t})$$

$$(b) \quad \zeta = 0.50 \quad \frac{x \omega_n}{v_0} = \frac{e^{-0.50 \omega_n t}}{0.865} \sin 0.865 \omega_n t \quad \text{Eq 2.6-17}$$

$$(c) \quad \zeta = 1.0 \quad \frac{x \omega_n}{v_0} = \omega_n t e^{-\omega_n t} \quad \text{Eq 2.6-23}$$

2-37 Solve on computer using RUNGA. With ω_n not given, the equation

$$\frac{d^2x}{dt^2} + 25 \frac{dx}{dt} + \omega_n^2 x = 0$$

can be rewritten for RUNGA as follows

$$\text{Let } T = \omega_n t \quad \text{and} \quad y = \frac{x \omega_n}{v_0}$$

$$\text{Then } x = \frac{v_0}{\omega_n} y, \quad \frac{dx}{dt} = \frac{v_0}{\omega_n} \frac{dy}{dt} = v_0 \frac{dy}{dT}$$

$$\frac{d^2x}{dt^2} = \frac{v_0}{\omega_n} \frac{d^2y}{dT^2} = \frac{v_0}{\omega_n} \frac{\omega_n^2}{\omega_n^2} \frac{d^2y}{dT^2} = v_0 \omega_n \frac{d^2y}{dT^2}$$

The original equation then becomes

$$v_0 \omega_n \left[\frac{d^2y}{dT^2} + 25 \frac{dy}{dT} + y \right] = 0$$

and $v_0 = 1.0$ for all computations.

Computer solutions for (a), (b) and (c) are:

$$(a) \quad \zeta = 2.0 \quad y_{\max} = \frac{\omega_n x_{\max}}{v_0} = 0.1953 \text{ (critical damping)}$$

$$(b) \quad \zeta = .50 \quad y_{\max} = 0.534 \text{ (oscillatory)}$$

$$(c) \quad \zeta = 1.0 \quad y_{\max} = 0.3666 \text{ (aperiodic)}$$

See Prob 2-36 for analytic solution:

» 237 Cent

Prob. 237(a)

Enter the value of the mass (kg) : 1

m =

1

Enter the value of the damping coefficient (N.s/m) : 4

c =

4

Enter the value of the spring constant (N/m) : 1

k =

1

Enter the value of the initial position (m) : 0

x1 =

0

Enter the value of the initial velocity (m/s) : 1

y1 =

1

ans =

Natural period

T =

6.2832

2-37 cont

Enter the value of time increment in seconds (< T as above) : .5

dt =

0.5000

Enter the value of the initial time in seconds (0) : 0

t1 =

0

Enter the value of the final time in seconds : 10

tf =

10

ans =

Time	Displ.	Vel.
------	--------	------

ans =

0	0	1.0000
0.5000	0.1667	0.2526
1.0000	0.1953	0.0360
1.5000	0.1856	-0.0235
2.0000	0.1667	-0.0368
2.5000	0.1471	-0.0371
3.0000	0.1290	-0.0339
3.5000	0.1130	-0.0301
4.0000	0.0988	-0.0264
4.5000	0.0864	-0.0231
5.0000	0.0756	-0.0203
5.5000	0.0661	-0.0177
6.0000	0.0578	-0.0155
6.5000	0.0506	-0.0136
7.0000	0.0442	-0.0119

2-3 f cont

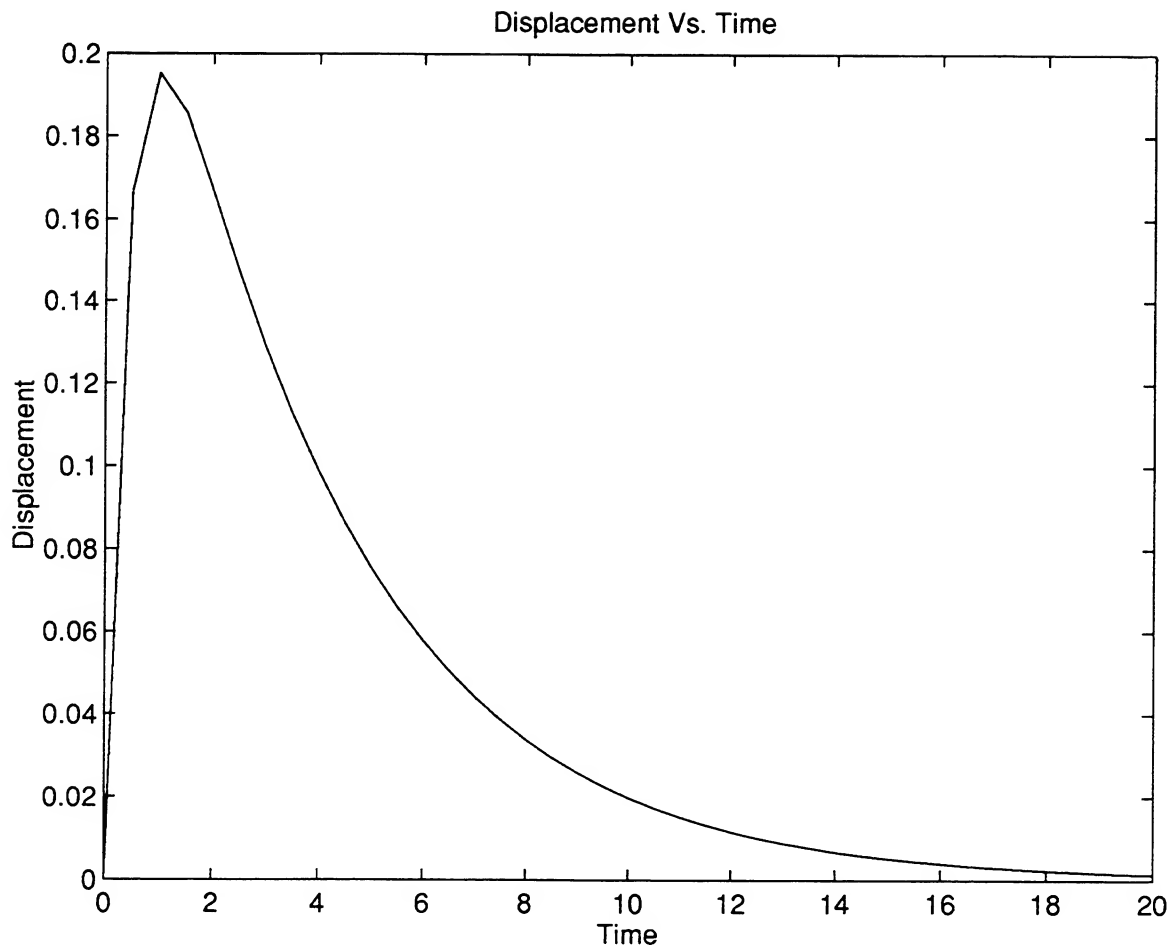
7.5000	0.0387	-0.0104
8.0000	0.0338	-0.0091
8.5000	0.0296	-0.0079
9.0000	0.0259	-0.0069
9.5000	0.0226	-0.0061
10.0000	0.0198	-0.0053

ans =

maximum amplitude

A =

0.1953



» 2 - 37 pent

2.37(b)

Enter the value of the mass (kg) : 1

m =

1

Enter the value of the damping coefficient (N.s/m) : 1

c =

1

Enter the value of the spring constant (N/m) : 1

k =

1

Enter the value of the initial position (m) : 0

x1 =

0

Enter the value of the initial velocity (m/s) : 1

y1 =

1

ans =

Natural period

T =

6.2832

2-37 cont

Enter the value of time increment in seconds ($< T$ as above) : .5

$\Delta t =$

0.5000

Enter the value of the initial time in seconds (0) : 0

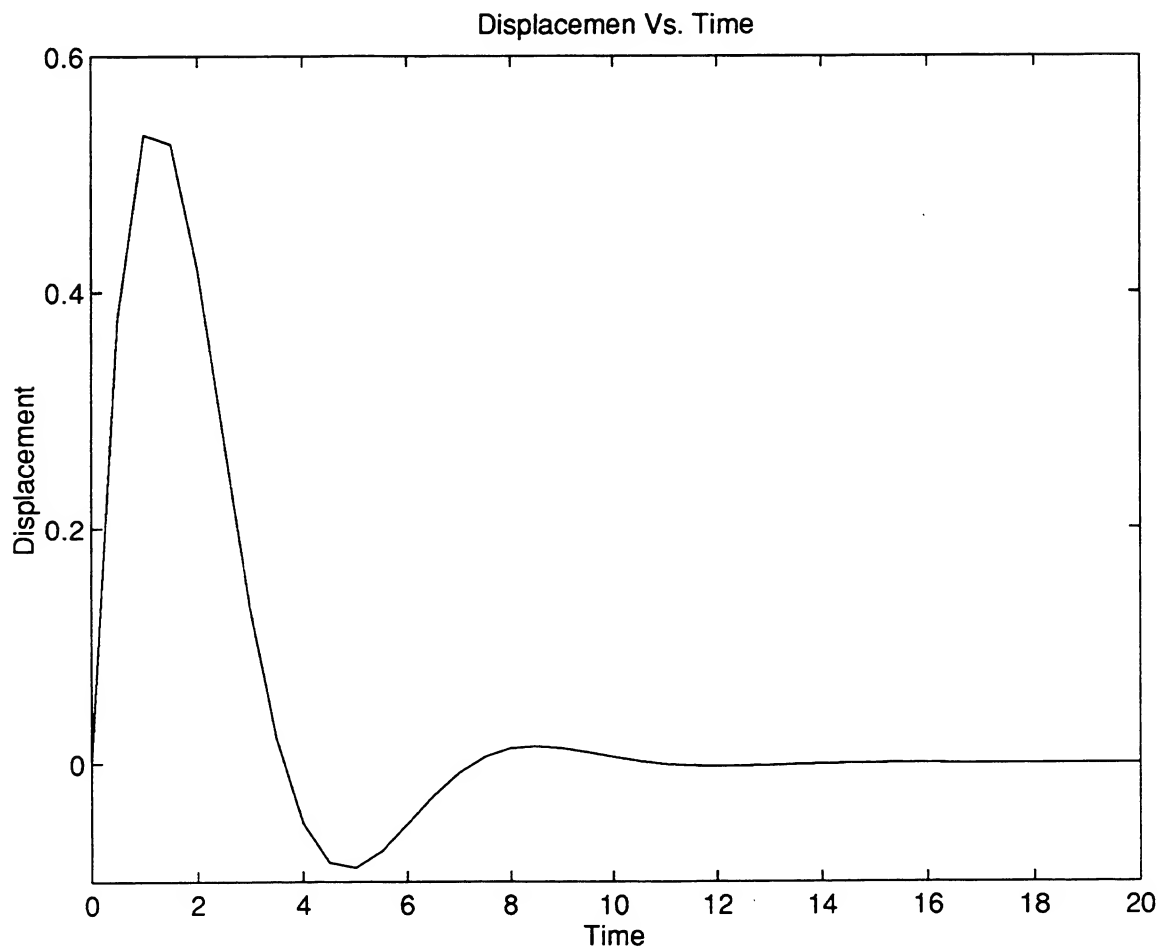
$t_1 =$

0

Enter the value of the final time in seconds : 20

$t_f =$

20



2-5+Cont

Time	Displ.	Vel.
0	0	1.0000
0.5000	0.3776	0.5182
1.0000	0.5340	0.1260
1.5000	0.5259	-0.1363
2.0000	0.4196	-0.2692
2.5000	0.2743	-0.2980
3.0000	0.1332	-0.2580
3.5000	0.0219	-0.1840
4.0000	-0.0499	-0.1036
4.5000	-0.0838	-0.0349
5.0000	-0.0882	0.0136
5.5000	-0.0739	0.0404
6.0000	-0.0510	0.0488
6.5000	-0.0272	0.0446
7.0000	-0.0076	0.0334
7.5000	0.0058	0.0202
8.0000	0.0128	0.0082
8.5000	0.0146	-0.0006
9.0000	0.0129	-0.0058
9.5000	0.0093	-0.0079
10.0000	0.0054	-0.0076
10.5000	0.0020	-0.0060
11.0000	-0.0005	-0.0038
11.5000	-0.0019	-0.0018
12.0000	-0.0024	-0.0002
12.5000	-0.0022	0.0008
13.0000	-0.0017	0.0012
13.5000	-0.0010	0.0013
14.0000	-0.0004	0.0011
14.5000	-0.0000	0.0007
15.0000	0.0003	0.0004
15.5000	0.0004	0.0001
16.0000	0.0004	-0.0001
16.5000	0.0003	-0.0002
17.0000	0.0002	-0.0002
17.5000	0.0001	-0.0002
18.0000	0.0000	-0.0001
18.5000	-0.0000	-0.0001
19.0000	-0.0001	-0.0000
19.5000	-0.0001	0.0000
20.0000	-0.0001	0.0000

2-37 cont

maximum amplitude

A =

0.5340

» prob 2.37(a)

Enter the value of the mass (kg) : 1

m =

1

Enter the value of the damping coefficient (N.s/m) : 2

c =

2

Enter the value of the spring constant (N/m) : 1

k =

1

Enter the value of the initial position (m) : 0

x1 =

0

Enter the value of the initial velocity (m/s) : 1

y1 =

1

ans =

2 - 37 cont

Natural period

T =

6.2832

Enter the value of time increment in seconds (< T as above) : .5

dt =

0.5000

Enter the value of the initial time in seconds (0) : 0

t1 =

0

Enter the value of the final time in seconds : 20

tf =

20

ans =

Time	Displ.	Vel.
------	--------	------

ans =

0	0	1.0000
0.5000	0.3021	0.3047
1.0000	0.3666	0.0016
1.5000	0.3337	-0.1103
2.0000	0.2699	-0.1344
2.5000	0.2047	-0.1225
3.0000	0.1491	-0.0992
3.5000	0.1055	-0.0752

2-37 cont

4.0000	0.0732	-0.0548
4.5000	0.0500	-0.0388
5.0000	0.0337	-0.0269
5.5000	0.0225	-0.0184
6.0000	0.0149	-0.0124
6.5000	0.0098	-0.0083
7.0000	0.0064	-0.0055
7.5000	0.0042	-0.0036
8.0000	0.0027	-0.0024
8.5000	0.0017	-0.0015
9.0000	0.0011	-0.0010
9.5000	0.0007	-0.0006
10.0000	0.0005	-0.0004
10.5000	0.0003	-0.0003
11.0000	0.0002	-0.0002
11.5000	0.0001	-0.0001
12.0000	0.0001	-0.0001
12.5000	0.0000	-0.0000
13.0000	0.0000	-0.0000
13.5000	0.0000	-0.0000
14.0000	0.0000	-0.0000
14.5000	0.0000	-0.0000
15.0000	0.0000	-0.0000
15.5000	0.0000	-0.0000
16.0000	0.0000	-0.0000
16.5000	0.0000	-0.0000
17.0000	0.0000	-0.0000
17.5000	0.0000	-0.0000
18.0000	0.0000	-0.0000
18.5000	0.0000	-0.0000
19.0000	0.0000	-0.0000
19.5000	0.0000	-0.0000
20.0000	0.0000	-0.0000

ans =

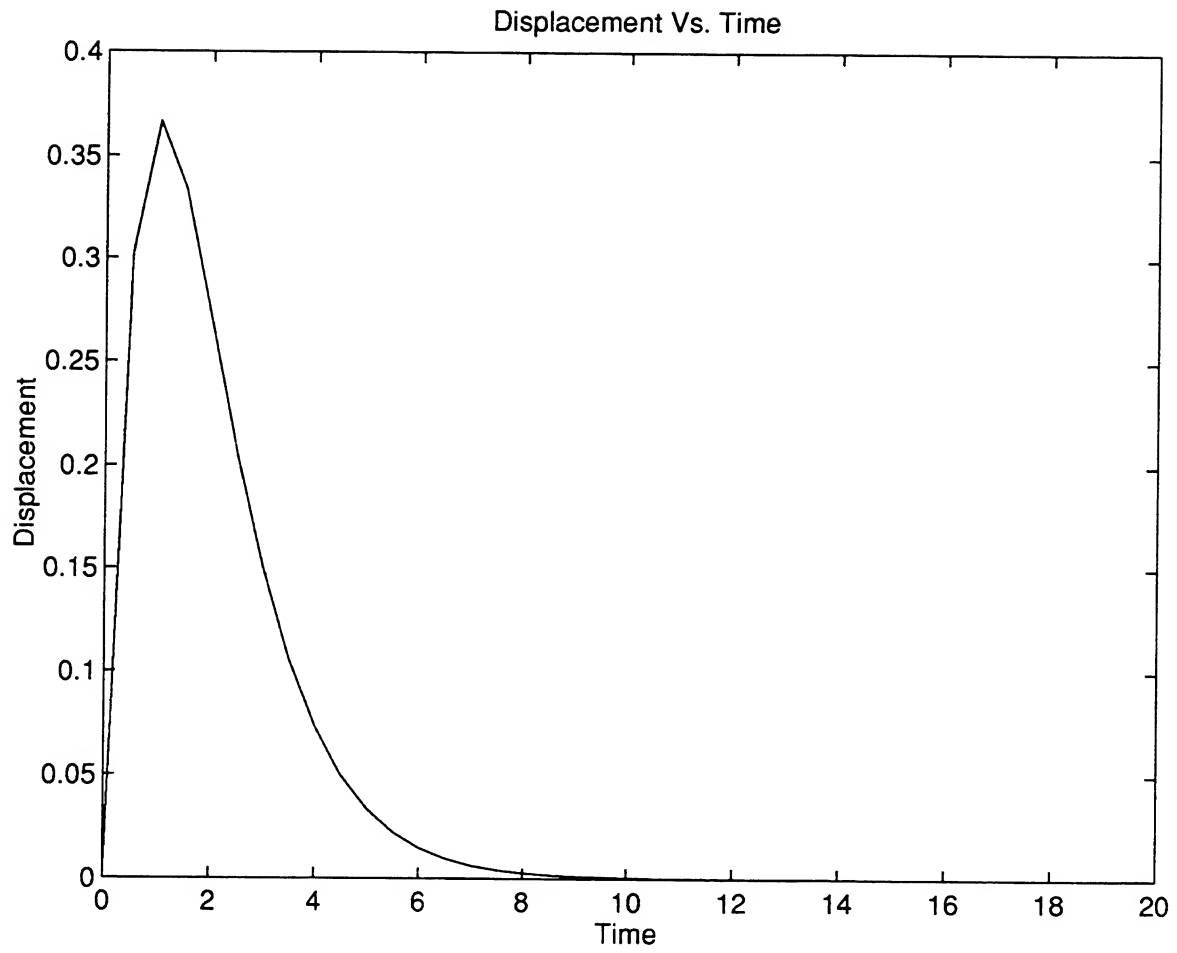
maximum amplitude

A =

0.3666

»

2-57 cont



2-38

$$\delta = \ln \frac{x_1}{x_2} = \ln \frac{1.00}{.980} = \ln 1.020408 = 0.0202$$

$$\zeta \approx \frac{\delta}{2\pi} = .003215$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1750}{2.267}} = 27.78 \approx \omega_d$$

$$c = 2m\omega_n\zeta = 2 \times 2.267 \times 27.78 \times .003215 = .405 \frac{Ns}{m}$$

2-39

$$(a) \quad \zeta = \frac{c}{2m} \sqrt{\frac{m}{k}} = \frac{12.43}{2 \times 4.534} \sqrt{\frac{4.534}{3500}} = 0.0493$$

$$(b) \quad \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 0.3101$$

$$(c) \quad \frac{x_n}{x_{n+1}} = e^{\delta} = (2.718)^{.3101} = 1.364$$

2-40

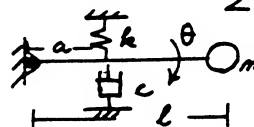
$$(a) \quad \zeta = \frac{c}{2\sqrt{mk}} = \frac{70}{2\sqrt{17.5 \times 7000}} = 0.10$$

$$(b) \quad f_d = \frac{1}{2\pi} \sqrt{1-\zeta^2} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{1-.01} \sqrt{\frac{7000}{17.5}} = 3.167 \text{ Hz}$$

$$(c) \quad \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = .6315$$

$$(d) \quad x_n/x_{n+1} = e^{.6315} = 1.874$$

2-41

$$\sum M_o = -ac(a\dot{\theta}) - 2k(a\theta) = mL^2\ddot{\theta}$$


$$\ddot{\theta} + \frac{c}{m} \left(\frac{a}{l}\right)^2 \dot{\theta} + \frac{k}{m} \left(\frac{a}{l}\right)^2 \theta = 0 \quad \text{let } \theta = e^{st}$$

$$s_{1,2} = -\frac{c}{2m} \left(\frac{a}{l}\right)^2 \pm \sqrt{\left(\frac{ca^2}{2ml^2}\right)^2 - \frac{k}{m} \left(\frac{a}{l}\right)^2}$$

$$\text{crit. damp. } \frac{c_c a^2}{2ml^2} = \frac{a}{l} \sqrt{\frac{k}{m}} \quad c_c = 2 \frac{l}{a} \sqrt{k m}$$

$$\omega_d = \frac{a}{l} \sqrt{\frac{k}{m} - \left(\frac{ca}{2ml^2}\right)^2} = \frac{a}{l} \sqrt{\frac{k}{m}} \sqrt{1 - \left(\frac{ca}{2l\sqrt{k m}}\right)^2} = \omega_n \sqrt{1-\zeta^2}$$

$$\therefore \omega_n = \frac{a}{l} \sqrt{\frac{k}{m}}, \quad \zeta = \frac{ca}{2l\sqrt{k m}} \quad \text{identify from } \ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

2-42

$$\Sigma M_o = ma^2 \ddot{\theta} = -kb^2 \theta - ca^2 \dot{\theta}$$

$$\ddot{\theta} + \frac{c}{m} \dot{\theta} + \frac{k}{m} \left(\frac{b}{a}\right)^2 \theta = 0$$

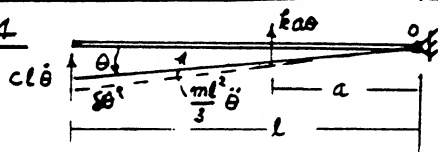
$$\therefore \omega_n = \frac{b}{a} \sqrt{\frac{k}{m}} \quad \omega_d = \sqrt{\frac{k}{m} \left(\frac{b}{a}\right)^2 - \left(\frac{c}{2m}\right)^2} \quad c_c = \frac{2b}{a} \sqrt{k m}$$

2-43

$$\delta = \ln \frac{1.0}{0.95} = \ln 1.0527 = .05129$$

$$\delta = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}} = .05129 \quad \zeta = .00816$$

2-44



$$\Sigma W = -\frac{ml}{3} \ddot{\theta} \delta \theta - c \dot{\theta} l \delta \theta - k a \theta \delta \theta = 0$$

$$\ddot{\theta} + \frac{3c}{m} \dot{\theta} + \frac{3k}{m} \left(\frac{a}{l}\right)^2 \theta = 0$$

$$\ddot{\theta} + 2\zeta \omega_n \dot{\theta} + \omega_n^2 \theta = 0$$

$$\therefore \omega_n = \frac{a}{l} \sqrt{\frac{3k}{m}}, \quad c_c = \frac{2a}{3l} \sqrt{3km}, \quad \zeta = \frac{3}{2} \frac{c}{m \omega_n}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = \frac{a}{l} \sqrt{\frac{3k}{m}} \sqrt{1 - \frac{9}{4} \left(\frac{c}{m \omega_n}\right)^2} = \frac{a}{l} \sqrt{\frac{3k}{m}} \sqrt{1 - \frac{3}{4km} \left(\frac{cl}{a}\right)^2}$$

2-45

$$\frac{W}{g} \ddot{x} + 2\mu A \dot{x} + kx = 0$$

$$\ddot{x} + \frac{2\mu A g}{W} \dot{x} + \frac{k g}{W} x = 0 \quad \therefore \gamma_1 = 2\pi \sqrt{\frac{W}{kg}}$$

$$f_2 = \frac{1}{\gamma_2} = \frac{1}{2\pi} \sqrt{\frac{kg}{W} - \left(\frac{\mu A g}{W}\right)^2} = \frac{1}{2\pi} \sqrt{\left(\frac{2\pi}{\gamma_1}\right)^2 - \left(\frac{\mu A g}{W}\right)^2}$$

square both sides

$$\left(\frac{2\pi}{\gamma_2}\right)^2 - \left(\frac{2\pi}{\gamma_1}\right)^2 = -\left(\frac{\mu A g}{W}\right)^2$$

$$\therefore \mu = \frac{2\pi W}{A g} \sqrt{\frac{\gamma_2^2 - \gamma_1^2}{\gamma_1^2 \gamma_2^2}} = \frac{2\pi W}{A g \gamma_1 \gamma_2} \sqrt{\gamma_2^2 - \gamma_1^2}$$

2-46

$$\omega_n = \sqrt{\frac{20,000 \times 32.2}{1200}} = 23.17 \text{ rad/s}$$

$$\frac{1}{2} m \dot{x}_{\max}^2 = \frac{1}{2} k x_{\max}^2 \quad \dot{x}_{\max} = 23.17 \times 4 = 92.66 \text{ ft/s}$$

$$\text{Eq. (2.3-19)} \quad x = e^{-\omega_n t} [0 + \omega_n x(0)]t + x(0) e^{-\omega_n t}$$

$$= e^{-\omega_n t} x(0) [1 + \omega_n t]$$

$$\frac{2}{12} = e^{-\omega_n t} 4 [1 + \omega_n t] \quad \text{or} \quad e^{-\omega_n t} [1 + \omega_n t] = 0.0417$$

solve by trial

$\omega_n t$	$e^{-\omega_n t}$	$e^{-\omega_n t} [1 + \omega_n t]$
4.90	.00745	.0439
4.96	.007017	.04182 ← close
4.97	.006947	.04147

$$\therefore \omega_n t = 4.96 \quad t = \frac{4.96}{23.17} = 0.214 \text{ s}$$

2-47

$$\omega_n = \sqrt{\frac{35000}{4.53}} = 87.89 \text{ rad/s}$$

$$\gamma = \frac{2\pi}{87.89} = .0715 \text{ s} \quad c_c = 2\sqrt{km} = 797.04$$

$$\zeta = .2197$$

$$\gamma_d = \sqrt{1 - \zeta^2} \gamma = .0697$$

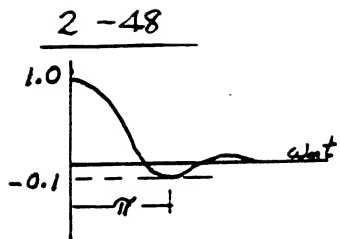
Eq. (2.6-16)

$$x = \frac{\dot{x}(0)}{\omega_n \sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \sqrt{1 - \zeta^2} \omega_n t$$

$$\text{at } x_{\max}, \sin \sqrt{1 - \zeta^2} \omega_n t \approx 1.0, \text{ also } \omega_n t = \pi/2$$

$$x = \frac{15.24}{87.89 \times .9756} e^{-.2197(\pi/2)} = .1259 \text{ m}$$

$$t = \frac{1}{4} \gamma_d = .0174 \text{ s}$$



Eq. (2.6-17) for $\dot{x}(0) = 0$

$$x = x(0) e^{-\zeta \omega_n t} \cos \sqrt{1-\zeta^2} \omega_n t$$

$$\text{at } \sqrt{1-\zeta^2} \omega_n t = \pi \quad \cos \sqrt{1-\zeta^2} \omega_n t = -1$$

$$-0.10 = 1 e^{-\zeta \omega_n t} (-1) \quad \text{solve by trial}$$

ζ	$\sqrt{1-\zeta^2}$	$\frac{-\pi}{\sqrt{1-\zeta^2}}$	$e^{\frac{-\pi \zeta}{\sqrt{1-\zeta^2}}}$
.50	.866		.1630
.59	.8074	-2.2957	.1007 ←
.60	.800	-2.3562	.0948

$$\therefore \zeta_1 = 0.59$$

$$\text{If } \zeta = \frac{1}{2} \zeta_1 = 0.295, \quad \sqrt{1-\zeta^2} = .9555$$

$$x_{\text{overshoot}} = 1 e^{\frac{-0.295 \pi}{.9555}} = 0.379 = 37.9\%$$

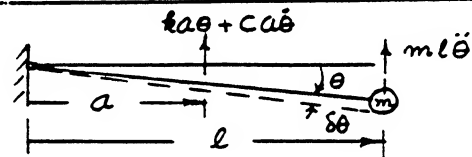
2-49(a) (Prob. 2-41 by V.W.)

$$\delta W = -m\ddot{\theta} \cdot l\delta\theta - (ka\theta + ca\dot{\theta}) a\delta\theta = 0$$

$$\ddot{\theta} + \frac{ca^2}{ml^2} \dot{\theta} + \frac{ka^2}{ml^2} \theta = 0$$

$$\ddot{\theta} + 2\zeta \dot{\theta} + \omega_n^2 \theta = 0 \quad \therefore \omega_n = \frac{a}{l} \sqrt{\frac{k}{m}}, \quad \zeta = \frac{1}{2} \frac{c}{m} \left(\frac{a}{l}\right)^2 \cdot \frac{l}{a} \sqrt{\frac{m}{k}} = \frac{ca}{2l\sqrt{k}m}$$

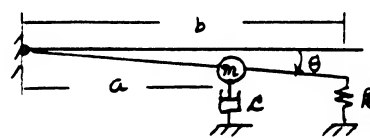
$$\omega_d = \frac{a}{l} \sqrt{\frac{k}{m}} \sqrt{1 - \frac{1}{4} \left(\frac{ca}{l\sqrt{k}m}\right)^2}$$



2-49(b) (Prob. 2-42 by V.W.)

$$\delta W = -ma\ddot{\theta} \cdot a\delta\theta - ca\dot{\theta} \cdot a\delta\theta - kb\theta \cdot b\delta\theta = 0$$

$$\therefore \ddot{\theta} + \frac{c}{m} \dot{\theta} + \frac{k}{m} \left(\frac{b}{a}\right)^2 \theta = 0$$

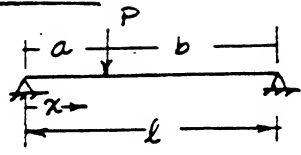


2-50

$$k_{\text{eff}} = (k_1 + k_2) \text{ in series with } k_3$$

$$= \frac{(k_1 + k_2) k_3}{k_1 + k_2 + k_3}$$

2-51



$$y(x) = \frac{Pbx}{6EI} (\ell^2 - x^2 - b^2)$$

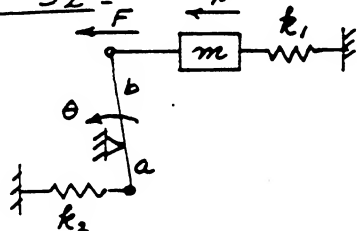
$$0 \leq x \leq a$$

$$\text{Let } x = \frac{\ell}{3}$$

$$y\left(\frac{\ell}{3}\right) = \frac{P \frac{\ell}{3} \cdot \frac{\ell}{3} \ell}{6EI} \left(\ell^2 - \frac{\ell^2}{9} - \frac{4}{9} \ell^2\right) = \frac{P\ell^3}{EI} \cdot \frac{4}{243}$$

$$\text{Flexibility} = \frac{y}{P} = \frac{4}{243} \frac{\ell^3}{EI} \quad \text{at } \frac{x}{\ell} = \frac{1}{3}$$

2-52



$$F = k_1 b \theta + \frac{a}{b} k_2 a \theta$$

$$x = b \theta$$

$$\therefore F = k_1 x + \left(\frac{a}{b}\right)^2 k_2 x$$

$$k_{\text{eff}} = \frac{F}{x} = k_1 + \left(\frac{a}{b}\right)^2 k_2$$

2-53

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}$$

2-54 Eq. (2.6-17)

$$x(t) = e^{-5\omega_n t} \left(\frac{5}{\sqrt{1-5^2}} \sin \sqrt{1-5^2} \omega_n t + \cos \sqrt{1-5^2} \omega_n t \right)$$

at $\omega_n t = 2\pi, 4\pi, 6\pi, \text{etc.}$

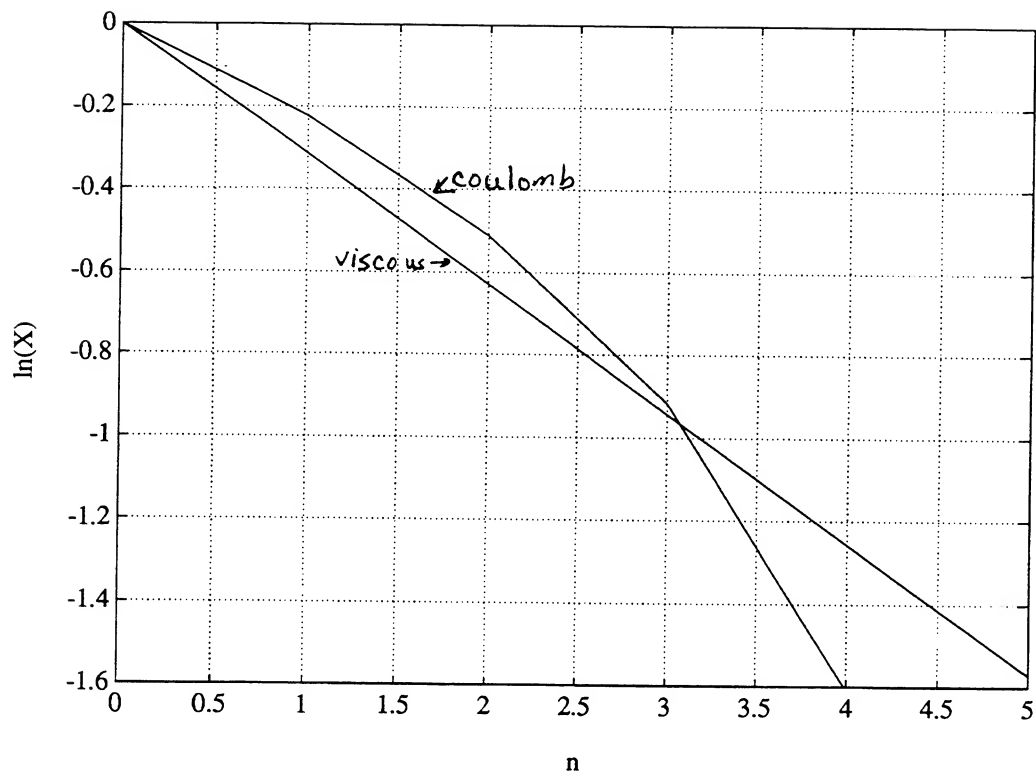
$$x(t) \approx e^{-5\omega_n t} (0 + 1)$$

n	$\omega_n t$	$e^{-.05\omega_n t}$	x_n
0	0	1.0	1.0
1	2π	.7304	.8
2	4π	.5335	.6
3	6π	.3896	.4
4	8π	.2845	.2
5	10π	.2078	0

For Coulomb friction

$$x_1 - x_2 = \frac{4F_d}{k} = \frac{4 \times .05k}{k} = .20$$

$$x_n = 1 - .2n$$



The two amplitudes are equal for $n \approx 3.1$

2-55

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I_0 \left(\frac{\dot{x}}{r} \right)^2$$

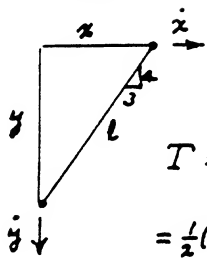
$$U = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 \left(a \frac{x}{r} + x \right)^2$$

$$\frac{d}{dt}(T+U) = \left(m_1 \ddot{x} + m_2 \ddot{x} + \frac{I_0}{r^2} \ddot{x} \right) \dot{x} + \left[k_1 + k_2 \left(1 + \frac{a}{r} \right)^2 \right] x \dot{x} = -c \dot{x} \dot{x} = \frac{dW}{dt}$$

$$\left(m_1 + m_2 + I_0/r^2 \right) \ddot{x} + \left[k_1 + k_2 \left(1 + \frac{a}{r} \right)^2 \right] x + c \dot{x} = 0$$

$$c_c = 2 \sqrt{k_{eff} m_{eff}} = 2 \sqrt{\left[k_1 + k_2 \left(1 + \frac{a}{r} \right)^2 \right] (m_1 + m_2 + I_0/r^2)}$$

2-56



$$x + y = l$$

$$x dx + y dy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\dot{y} = -\frac{x}{y} \dot{x}$$

$$T = \frac{1}{2}(ml)\frac{l}{3}\left(\frac{\dot{x}}{l}\right)^2 + \frac{1}{2}(ml)\left[\left(\frac{\dot{x}}{2}\right)^2 + \left(\frac{\dot{y}}{2}\right)^2\right] + \frac{1}{2}(ml)\frac{l^2}{12}\left[\frac{4\dot{x}}{5l} + \frac{3\dot{y}}{5l}\right]^2 + \frac{1}{2}M\dot{y}^2$$

$$= \frac{1}{2}(ml)\left\{\frac{l^2}{3}\left(\frac{\dot{x}}{l}\right)^2 + \left[\left(\frac{\dot{x}}{2}\right)^2 + \left(\frac{3\dot{x}}{8}\right)^2\right] + \frac{l^2}{12}\left[\frac{4}{5}\frac{\dot{x}}{l} + \frac{9}{20}\frac{\dot{x}}{l}\right]^2 + \frac{1}{2}M\left(\frac{3\dot{x}}{4}\right)^2\right\}$$

$$= \frac{1}{2}(ml)\left[\frac{1}{3} + \frac{1}{4} + \frac{9}{64} + \frac{1}{12}\left(\frac{16}{25} + \frac{81}{400} + \frac{72}{100}\right)\right]\dot{x}^2 + \frac{1}{2}M\frac{9}{16}\dot{x}^2$$

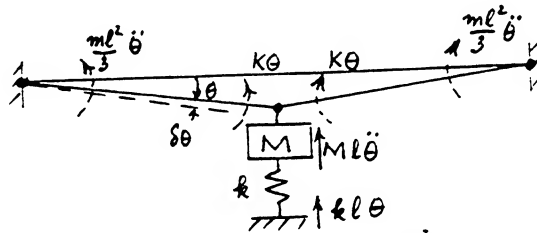
$$= \frac{1}{2}\left[(.854 ml) + .5625 M\right]\dot{x}^2$$

$$U = \frac{1}{2}k y^2 = \frac{1}{2}k \frac{9}{16} x^2$$

$$\frac{d}{dt}(T+U) = -c \frac{2}{3} l \frac{\dot{x}}{l} = -\frac{2}{3} c \dot{x}$$

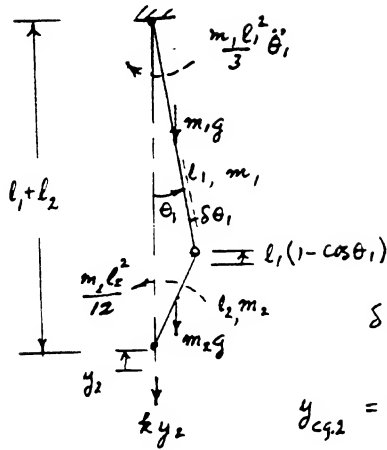
$$(0.8541 ml + .5625 M) \ddot{x} + .5625 k x + \frac{2}{3} c \dot{x} = 0$$

2-57



$$\delta W = [-Ml\ddot{\theta} l \delta\theta - k l \theta l \delta\theta] - 2K\theta \delta\theta - 2 \frac{ml^2}{3} \ddot{\theta} \delta\theta = 0$$

$$(Ml^2 + \frac{2}{3} ml^2) \ddot{\theta} + (kl^2 + 2K) \theta = 0$$



Accel of c.g. in vertical direction ≈ 0
but work done by gravity is not zero

$$\delta y_{cg,1} = \frac{l_1}{2} \sin \theta_1 \delta \theta_1 \approx \frac{l_1}{2} \theta_1 \delta \theta_1$$

$$y_2 = l_1(1 - \cos \theta_1) + l_2(1 - \cos \theta_2)$$

$$l_1 \theta_1 \approx l_2 \theta_2, \quad \theta_2 = \frac{l_1}{l_2} \theta_1, \quad \therefore \delta \theta_2 = \frac{l_1}{l_2} \delta \theta_1$$

$$\delta y_2 = l_1(\theta_1 + \frac{l_1}{l_2} \theta_1) \delta \theta_1 = l(1 + \frac{l_1}{l_2}) \theta_1 \delta \theta_1$$

$$\begin{aligned} y_{cg,2} &= l_1(1 - \cos \theta_1) + \frac{l_2}{2}(1 - \cos \theta_2) \quad \therefore \delta y_{cg,2} = l_1 \theta_1 \delta \theta_1 + \frac{l_2}{2} \theta_2 \delta \theta_2 \\ &= l_1 \theta_1 \delta \theta_1 + \frac{l_2}{2} \frac{l_1}{l_2} \theta_1 \frac{l_1}{l_2} \delta \theta_1 \\ &= l_1(1 + \frac{l_1}{2l_2}) \theta_1 \delta \theta_1 \end{aligned}$$

$$\begin{aligned} \delta W &= -(\frac{m_1 l_1^2}{3}) \ddot{\theta}_1 \delta \theta_1 - (\frac{m_2 l_2^2}{12} \ddot{\theta}_2) \delta \theta_2 - m_1 g \frac{l_1}{2} \theta_1 \delta \theta_1 - m_2 g l_1(1 + \frac{l_1}{2l_2}) \theta_1 \delta \theta_1 \\ &\quad - k[l_1(1 - \cos \theta_1) + l_2(1 - \cos \theta_2)] l(1 + \frac{l_1}{l_2}) \theta_1 \delta \theta_1 = 0 \end{aligned}$$

$$\text{sub. } \ddot{\theta}_2 = \frac{l_1}{l_2} \ddot{\theta}_1, \quad \delta \theta_2 = \frac{l_1}{l_2} \delta \theta_1, \quad \theta_2 = \frac{l_1}{l_2} \theta_1$$

$$[\frac{m_1 l_1^2}{3} + \frac{m_2 l_2^2}{12} (\frac{l_1}{l_2})^2] \ddot{\theta}_1 + [m_1 g \frac{l_1}{2} + m_2 g l_1(1 + \frac{l_1}{2l_2})] \theta_1 = 0$$

(Spring force is 2nd order infinitesimal)

2-59

With the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, the solution must be considered for each half cycle. During the first half cycle, with m moving from right to left, $\dot{x}(t)$ is negative so that $\text{sgn}(\dot{x})$ is positive and the equations of motion are;

$$x(t) = A \sin \omega_n t + B \cos \omega_n t + \frac{\mu F}{k}$$

$$\dot{x}(t) = \omega_n A \cos \omega_n t - \omega_n B \sin \omega_n t$$

At $t=0$

$$x(0) = x_0 = B + \frac{\mu F}{k} \quad \therefore B = (x_0 - \frac{\mu F}{k})$$

$$\dot{x}(0) = 0 = \omega_n A \quad \therefore A = 0$$

and the general solution becomes

$$x(t) = (x_0 - \frac{\mu F}{k}) \cos \omega_n t + \frac{\mu F}{k}$$

$$\dot{x}(t) = -\omega_n (x_0 - \frac{\mu F}{k}) \sin \omega_n t$$

For the second half cycle \dot{x} is positive and the friction force is negative. The new equation must now be written with two other constants C and D

$$x(t) = C \sin \omega_n t + D \cos \omega_n t - \frac{\mu F}{k}$$

$$\dot{x}(t) = \omega_n C \cos \omega_n t - \omega_n D \sin \omega_n t$$

We again measure time $t=0$ at the beginning of the 2nd half cycle, with new initial conditions found from the previous equations as $x(0) = (-x_0 + 2\frac{\mu F}{k})$ and $\dot{x}(0) = 0$

$$\therefore D = -x_0 + 2\frac{\mu F}{k} \quad \text{and} \quad C = 0$$

\therefore The new equations for the second half cycle

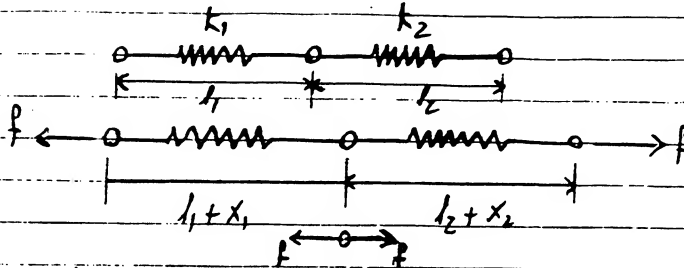
$$x(t) = (-x_0 + 2\frac{\mu F}{k}) \cos \omega_n t - \frac{\mu F}{k}$$

$$\dot{x}(t) = -\omega_n (-x_0 + 2\frac{\mu F}{k}) \sin \omega_n t$$

2-60

unstretched

stretched



In the stretch position the total displacement of the two springs is $x_1 + x_2$. Since each spring exhibits the same force F , we have

$$x_{eq} = x_1 + x_2$$

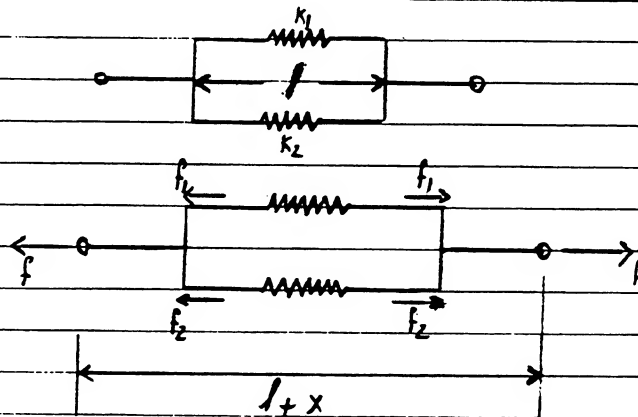
$$\therefore \frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2} \Rightarrow k_{eq} = \frac{1}{1/k_1 + 1/k_2}$$

where, x_{eq} and k_{eq} are the equivalent displacement and stiffness of the series combination.

2-61

unstretched

stretched

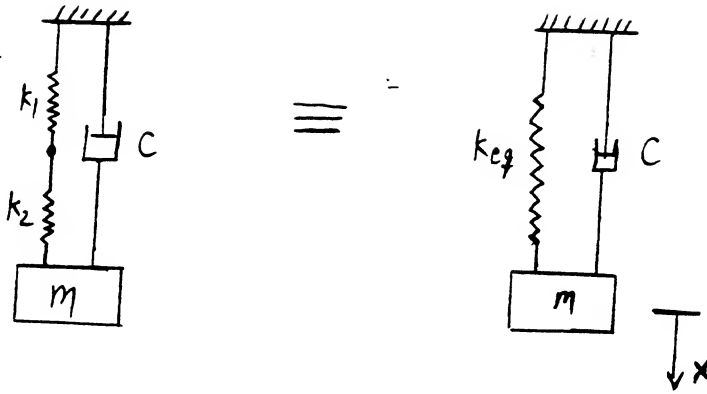


$$F = F_1 + F_2$$

$$\therefore k_{eq} x = k_1 x + k_2 x \Rightarrow k_{eq} = k_1 + k_2$$

where, k_{eq} is the equivalent stiffness of the parallel combination.

2.62



Equation of motion

$$m\ddot{X} = -k_{eq}X - C\dot{X} \Rightarrow m\ddot{X} + C\dot{X} + k_{eq}X = 0,$$

where, $k_{eq} = \frac{1}{1/k_1 + 1/k_2}$ (see problem 2.60) is the

effective spring constant for the system.

$$\underline{3-1} \quad x_{res.} = \frac{F}{c\omega_n} = \frac{F\gamma}{2\pi c}$$

$$c = \frac{F\gamma}{2\pi x_{res.}} = \frac{24.46 \times 0.20}{2\pi \cdot 1.27 \times 10^{-2}} = 61.3 \quad \frac{Ns}{m}$$

3-2

$$\frac{x_{undamped}}{x_{damped}} = \sqrt{\frac{(\omega_n^2 - \omega^2)^2 + (c\omega/m)^2}{(\omega_n^2 - \omega^2)^2}} = R$$

$$\omega_n = \frac{2\pi}{\gamma} = \frac{6.283}{.20} = 31.416 \quad \omega = 8\pi = 25.13$$

$$\frac{c\omega}{m} = \frac{61.3 \times 8\pi}{1.95} = 790.1$$

$$R = \sqrt{\frac{(31.4^2 - 25.13^2)^2 + (790.1)^2}{(31.4^2 + 25.13^2)^2}} = 2.44$$

$$\underline{3-3} \quad \delta = \ln 4.2 = 1.435 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Square & solve for ζ^2

$$2.0595(1 - \zeta^2) = 39.478 \zeta^2$$

$$\zeta^2 = \frac{2.059}{41.538} = .0496, \quad \zeta = .223$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{1.80} = \omega_n \sqrt{1 - \zeta^2} \quad \omega_n = \frac{2\pi}{1.80 \sqrt{1 - .0496}} = 3.5806$$

$$\omega = 3 \quad \frac{\omega}{\omega_n} = .8378$$

$$\begin{aligned} \text{Eq(3.1-7)} \quad X &= \frac{F_0/k}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + [2\zeta \frac{\omega}{\omega_n}]^2}} = \frac{2/525}{\sqrt{[1 - .702]^2 + .1396}} \\ &= \frac{2/525}{.4779} = .00797 \text{ m} = .797 \text{ cm} \end{aligned}$$

Eq(3.1-8)

$$\phi = \tan^{-1} \frac{2\zeta(\frac{\omega}{\omega_n})}{1 - (\frac{\omega}{\omega_n})^2} = \tan^{-1} \frac{.446 \times .8378}{.29801} = \tan^{-1} 1.2538 = 51.43^\circ$$

$$m\ddot{x}_1 = -c\dot{x}_1 + k(x_2 - x_1)$$

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = kX_2 \sin \omega t$$

Replace excitation by $kX_2 e^{i\omega t}$, then $x = X_1 e^{i(\omega t - \phi)}$

$$= X_1 e^{-i\phi} e^{i\omega t} = \bar{X}_1 e^{i\omega t}$$

$$[(k - m\omega^2) + i\omega c] \bar{X}_1 e^{i\omega t} = kX_2 e^{i\omega t}$$

$$\bar{X}_1 = \frac{kX_2}{(k - m\omega^2) + i\omega c} = \frac{kX_2 e^{-i\phi}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\therefore X_1 = \frac{kX_2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad \phi = \tan^{-1} \frac{c\omega}{k - m\omega^2}$$

3-8

$$m\ddot{x} = c(\dot{y} - \dot{x}) + kx$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y}$$

$$\text{let } y = Y e^{i\omega t}$$

$$x = X e^{i(\omega t - \phi)} = X e^{-i\phi} e^{i\omega t}$$

$$\dot{y} = i\omega Y e^{i\omega t}$$

$$= \bar{X} e^{i\omega t}$$

$$(k - m\omega^2 + i\omega c) \bar{X} = i\omega Y c$$

$$\bar{X} = \frac{i\omega Y c}{k - m\omega^2 + i\omega c} = \frac{i\omega Y c e^{-i\gamma}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = X e^{-i\phi}$$

$$X e^{-i\phi} = \frac{\omega Y c}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} i e^{-i\gamma} \quad \therefore e^{-i\phi} = i e^{-i\gamma} = e^{-i(\gamma - \frac{\pi}{2})}$$

$$X = \frac{\omega Y c}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \quad \phi = \gamma - \frac{\pi}{2}, \quad \tan \gamma = \frac{c\omega}{k - m\omega^2}$$

3-4 Square Eq. (3.1-7)

$$\left(\frac{X}{X_0}\right)^2 = \frac{1}{(1-r)^2 + 4\zeta^2 r}$$

where $X_0 = F/k$
 $r = (\omega/\omega_n)^2$

$$\frac{\partial}{\partial r} \left(\frac{X}{X_0}\right)^2 = \frac{2(1-r) - 4\zeta^2}{(\text{denom.})^2} = 0 \quad \therefore r = 1 - 2\zeta^2 = (\omega/\omega_n)_p^2$$

$$\left(\frac{\omega}{\omega_n}\right)_p = \sqrt{1 - 2\zeta^2}$$

3-5

At resonance $\frac{\omega}{\omega_n} = 1.0$, $\frac{X}{X_0} = \frac{1}{2\zeta} = \frac{.58}{X_0}$

When $\frac{\omega}{\omega_n} \neq 1.0$

$\therefore X_0 = 1.16\zeta$

$$\frac{X}{X_0} = \frac{1}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} = \frac{.46}{X_0}$$

Square & solve for ζ^2

$$\frac{1}{[1-.64]^2 + [2\zeta]^2 \cdot .64} = \frac{.2116}{(1.16\zeta)^2}$$

$$6.359 \zeta^2 = .1296 + 2.560 \zeta^2$$

$$\zeta^2 = .0341$$

$$\zeta = .1847$$

3-6

Eq. 3.1-17 $H(\omega) = \frac{1-r^2}{[1-r^2]^2 + [2\zeta r]^2} - i \frac{2\zeta r}{[1-r^2]^2 + [2\zeta r]^2}$
 $= R + i I$

%This is the function file for the imaginary part of prob. 3-6

function [imapp]=imp(r)

zeta=0.01;

imapp=-(2*zeta.*r)./((1-r.*r).^2+(2*zeta.*r).^2);

end

%This is the function file for the real part of prob. 3-6

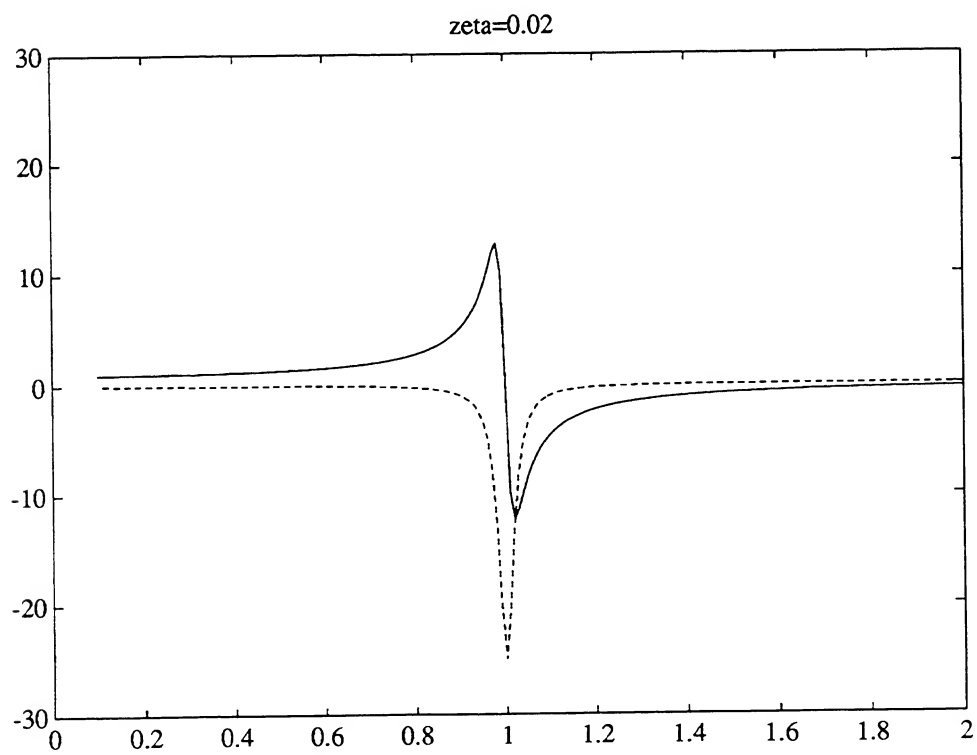
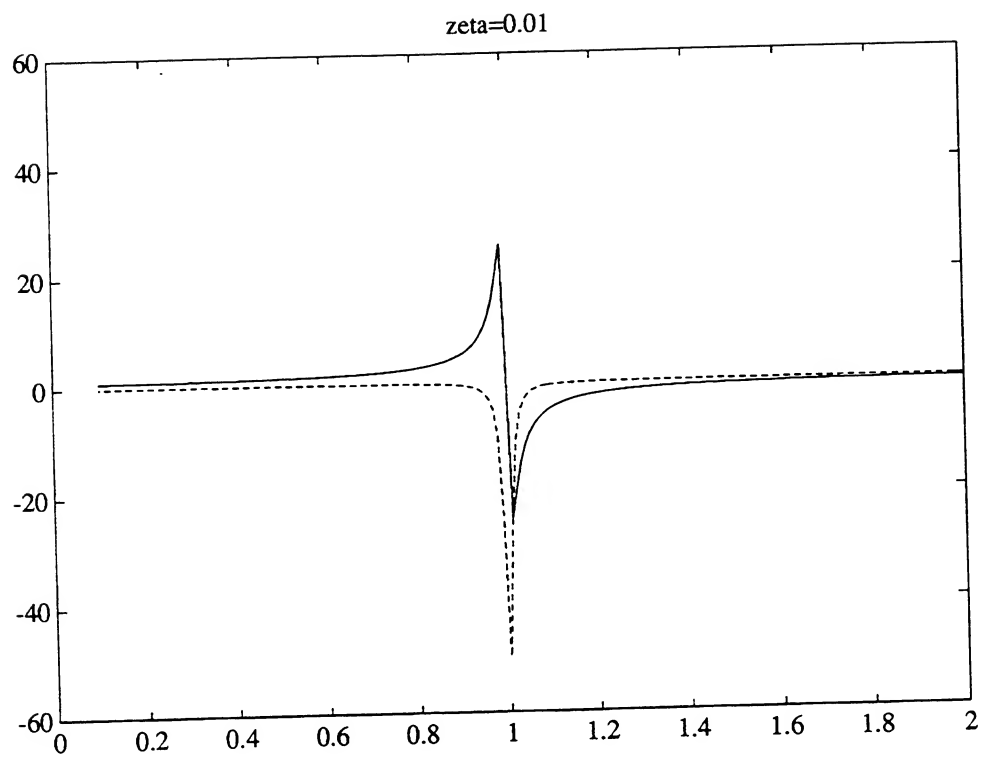
function [realp]=rp(r)

zeta=.01;

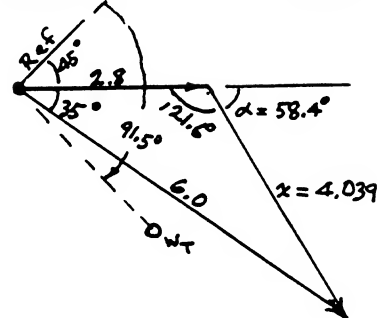
realp=(1-r.*r)./((1-r.*r).^2+(2*zeta.*r).^2);

end

prob 3-6 cont



3-9



Law of Cosines

$$\begin{aligned} x^2 &= 2.8^2 + 6.0^2 - 2 \times 2.8 \times 6.0 \times \cos 35^\circ \\ &= 7.84 + 36.0 - 33.60 \times .8192 = 16.316 \\ x &= 4.039 \end{aligned}$$

Law of Sines

$$\frac{4.039}{\sin 35^\circ} = \frac{6.0}{\sin \alpha}$$

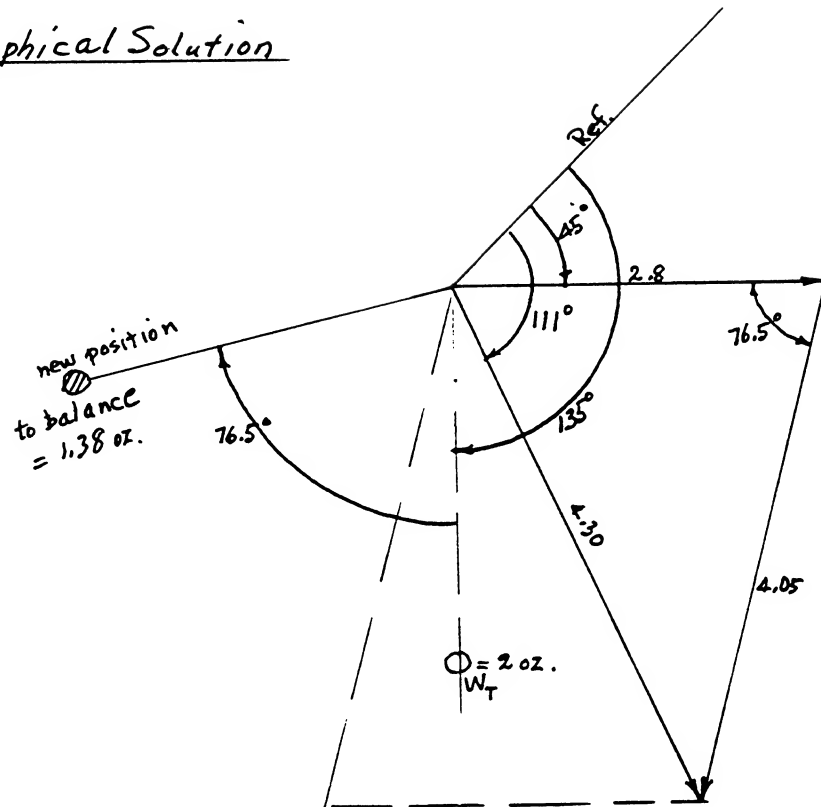
$$\begin{aligned} \sin \alpha &= \frac{6}{4.039} \times .5736 = .8521 \\ \alpha &= 58.44^\circ \\ 180 - \alpha &= 121.6^\circ \end{aligned}$$

$$2.03 \times \frac{2.8}{4.036} = 1.386 \text{ oz.}$$

To balance, rotate the vector $x = 4.039$, 121.6° clockwise and multiply by $\frac{2.8}{4.036}$. The position of the balancing wt is found by rotating the trial weight 121.6° c.w. from the position (of 91.5° from ref.) and reducing its value to 1.386 oz.

3-10

Graphical Solution



Original unbalance 2.8 @ 45° c.w. from Ref.

With trial weight $W_T = 2$ @ 135° c.w. " " new reading = 4.30 @ 111° c.w.

Resultant = 4.05 @ 76.5° From horizontal

To balance rotate W_T 76.5° and reduce to $\frac{2.8}{4.05} \times 2 = 1.38$ oz.

Students should check with prob. 3-9

3-11

From Eq. 3.2-5, we have

$$\tan \phi = \frac{25 \left(\frac{\omega}{\omega_n} \right)}{1 - \left(\frac{\omega}{\omega_n} \right)^2} = \frac{2 \times 0.10 \times \frac{6}{9}}{1 - \left(\frac{6}{9} \right)^2} = \frac{0.1333}{1 - 0.444}$$

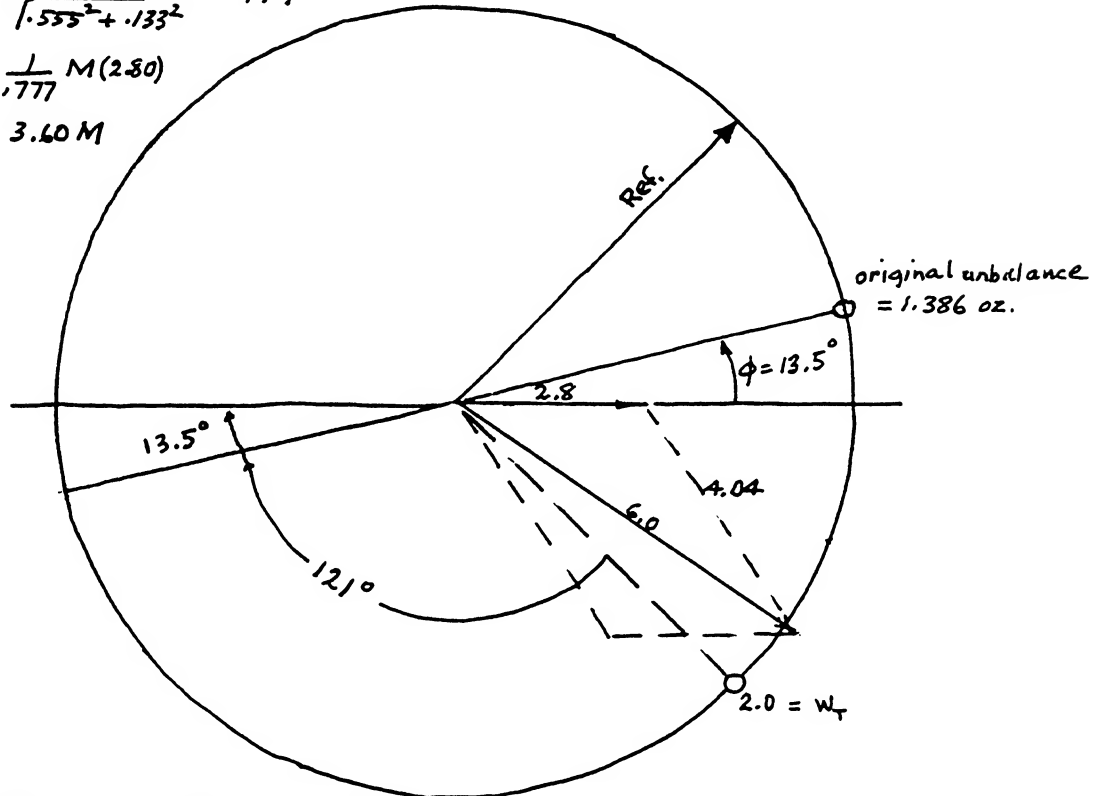
$$= 0.2399 \quad \therefore \phi = \underline{\underline{13.49^\circ}}$$

Eq. 3.2-4

$$\frac{MX}{me} = \frac{.444}{\sqrt{.555^2 + .133^2}} = .777$$

$$me = \frac{1}{.777} M(2.80)$$

$$= 3.60 M$$



Original unbalance = 2.8 = X

$$MX = .777 me \quad \therefore (me)_1 = 3.60 M$$

2 oz trial wt w_T resulted in $X = 4.04$

$$M 4.04 = .777 (me)_2 \quad \therefore (me)_2 = 5.199 M$$

$$\frac{(me)_2}{(me)_1} = \frac{5.199}{3.60} = 1.444 \quad \therefore (me)_2 = 1.444 \times 1.386 = 2.0 \text{ oz}$$

$$= w_T$$

3-12

The method used to balance the single disk can be modified to balance the long rotor shown in Fig P3-12 (a).

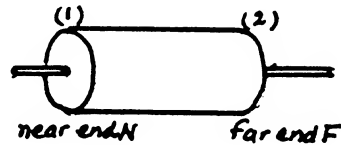


Fig P3-12 (a)

- (1) First measure the original unbalance vectors w.r.t. a fixed reference on the rotor at the near and far ends as N and F respectively.
- (2) Next, a trial weight W_{t1} is added to the near end and the new vectors N_1 and F_1 , shown in Fig P3-12 (b) are recorded with W_{t1} . The dotted vectors A and αA are due to W_{t1} alone,

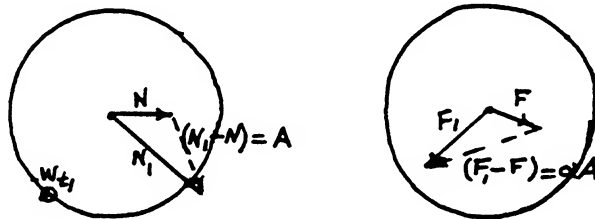


Fig P3-12 (b)

and since they are proportional to W_{t1} , they can be related to each other by a complex influence coefficient, or operator α .

- (3) Next, the trial weight W_{t1} is removed and a new trial weight W_{t2} is added to the far end. Measurements for N_2 and F_2 are then recorded, as shown in Fig P3-12 (c).

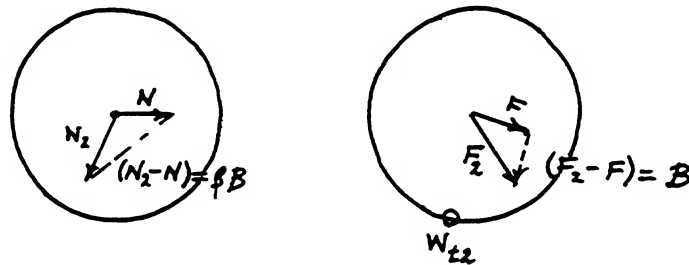


Fig P 3-12 (c)

3-12 Cont.

Again, the dotted vectors are proportional to W_{t2} , and they are related to each other by another complex operator β .

- (4) As in the single disk problem, the two dotted vectors for the near end must be rotated and combined to cancel the original unbalance N . Also the final balance weights W_{b1} and W_{b2} must be related to the trial weights W_{t1} and W_{t2} by two additional complex operators θ and ϕ .

$$W_{b1} = \theta W_{t1}$$

$$W_{b2} = \phi W_{t2}$$

Thus to balance the long rotor, the following two equations must be satisfied

$$\theta A + \phi(\beta B) = -N$$

$$\phi B + \theta(\alpha A) = -F$$

Solving these for θ and ϕ , we obtain

$$\theta = \frac{\beta F - N}{(1 - \alpha\beta) A}$$

$$\phi = \frac{\alpha N - F}{(1 - \alpha\beta) B}$$

In these equations, N and F are the original unbalance vectors, and α and β are obtained by the division

$\alpha = \frac{\alpha A}{A}$ and $\beta = \frac{\beta B}{B}$. With θ and ϕ known, the balancing weights W_{b1} and W_{b2} are found.

3-13
 (a) $\varphi = 90^\circ \therefore \text{resonance}$ $\omega_m = \omega = \frac{900 \times 2\pi}{60} = 2\pi f_m$
 $f_m = 15 \text{ cps} = 900 \text{ cpm}$

(b) Eq. (3.2-4) at resonance

$$\zeta = \frac{cm}{2MX} = \frac{2 \times 0.0921}{2 \times 181.4 \times 21.6 \times 10^{-3}} = 0.0118 \times 2$$

(c) At 1200 rpm $\frac{\omega}{\omega_m} = \frac{1200}{900} = 1.333$

Eq. (3.2-4)

$$X = \frac{me}{M} \sqrt{\frac{1.333^2 \times 2}{[1 - 1.333^2]^2 + [2 \times 0.0118 \times 1.333]^2}}$$

$$= \frac{2 \times 0.0921}{181.4} \times \frac{1.777}{.78023} = .002314 \text{ m} = .2314 \text{ cm}$$

(d) Eq. (3.2-5) $\varphi = \tan^{-1} \frac{2 \times .0235 \times 1.333}{1 - 1.333^2} = \tan^{-1}(-.0805)$

$$= 180^\circ - 4.61^\circ = 175.4^\circ$$

3-14

$$M\ddot{x} + c\dot{x} + kx = (me\omega^2)\sin\omega t$$

$$\text{Let } (me\omega^2)\sin\omega t = F e^{i\omega t}$$

$$\text{then } x = X e^{i(\omega t - \varphi)} = \bar{X} e^{-i\varphi} e^{i\omega t} = \bar{X} e^{i\omega t}$$

$$(-\omega^2 M + ic\omega + k)\bar{X} e^{i\omega t} = F e^{i\omega t}$$

$$\bar{X} = \frac{F}{(k - \omega^2 M) + i(c\omega)}$$

3-15

$$\omega = \frac{1200}{60} = 20 \text{ rps} = 20 \times 2\pi \text{ rad/s.}$$

$$\omega_n = 18 \text{ cps} = 18 \times 2\pi \text{ rad/s. } \therefore \frac{\omega}{\omega_n} = 1.111$$

Eq. 3.2-6

$$x(t) = X_1 e^{-\zeta \omega_n t} \sin(\sqrt{1-\zeta^2} \omega_n t - \phi_1) + \frac{m e \omega^2 \sin(\omega t - \varphi)}{\sqrt{(k - m \omega^2)^2 + (c \omega)^2}}$$

transient steady state

at $t=0$, $x(0) = 0$

$$0 = -X_1 \sin \phi_1 + \frac{(m e \omega^2 / k) \sin(-\varphi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}} \quad (1)$$

at $t=0$, $\dot{x}(0) = 0$

$$\dot{x}(t) = X_1 e^{-\zeta \omega_n t} \left[\sqrt{1-\zeta^2} \omega_n \cos(\sqrt{1-\zeta^2} \omega_n t - \phi_1) - \zeta \omega_n \sin(\sqrt{1-\zeta^2} \omega_n t - \phi_1) \right] + \frac{(m e \omega^2 / k) \omega \cos(\omega t - \varphi)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}}$$

$$\therefore \dot{x}(0) = 0 = X_1 [\omega_n \sqrt{1-\zeta^2} \cos \phi_1 + \zeta \omega_n \sin \phi_1] + \frac{(m e \omega^2 / k) \omega \cos \varphi}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}} \quad (2)$$

from (1) $X_1 \sin \phi_1 = \frac{-(m e \omega^2 / k) \sin \varphi}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}}$

from (2)

$$X_1 \omega_n \sqrt{1-\zeta^2} \cos \phi_1 + \zeta \omega_n \left[\frac{-(m e \omega^2 / k) \sin \varphi}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}} + \frac{(m e \omega^2 / k) \omega \cos \varphi}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}} \right] = 0$$

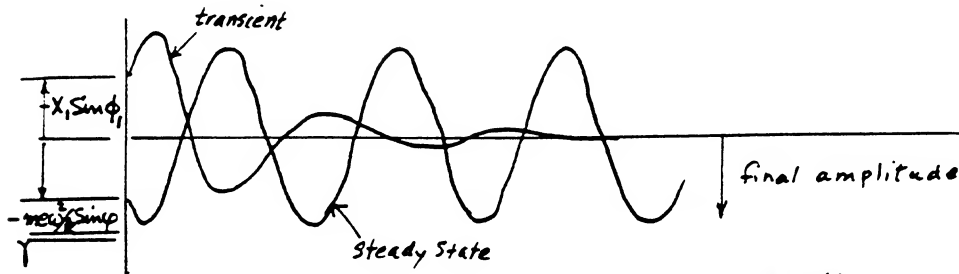
dividing

$$\therefore \tan \phi_1 = \frac{-\sqrt{1-\zeta^2} \sin \varphi}{\zeta \sin \varphi - \frac{\omega}{\omega_n} \cos \varphi} \quad (3)$$

3-15 Cont.

at $t = \infty$, transient term = 0 due to $e^{-5\omega_n t} \rightarrow 0$

$$\therefore \text{final ampl.} = x(\infty) = \frac{(m\omega/k) \sin(\omega t - \phi)}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + [25 \frac{\omega}{\omega_n}]^2}}$$



$$\text{phase } \phi \text{ can then be solved from } \tan \phi = \frac{25 \frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2} = \frac{.2(1.111)}{1 - (1.111)^2}$$

$$= -.94755 \quad \therefore \phi = -43.457^\circ \quad \text{Eq. (3.2-5)}$$

Then ϕ_1 solved from (3) $\tan \phi_1 = -.7820$, $\phi_1 = 142^\circ$

Then solve for X_1 and sub. into Eq. (3.2-6) for build up eq.

3-16 Lowest critical speed = fundamental freq. of lateral vibr.

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \text{where } k = \frac{48EI}{l^3}, \quad I = \frac{\pi d^4}{64}$$

$$m = m_{\text{disk}} + 0.486 m_{\text{shaft}}$$

$$k = \frac{48(29 \times 10^6) \pi (\frac{1}{2})^4}{(24)^3 \times 64} = 309. \text{ lb/in}$$

$$.486 m_{\text{shaft}} = \frac{.486 (.283) \frac{\pi}{4} (\frac{1}{2})^2 (24)}{386} = \frac{.648}{386}$$

$$m = \frac{10}{386} + \frac{.648}{386} = \frac{10.65}{386}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{309 \times 386}{10.65}} = 16.84 \text{ cps} = 1028 \text{ rpm}$$

3-17

$$W = 10 \text{ lb} = 44.48 \text{ N} \quad m = \frac{44.48}{9.81} = 4.534 \text{ kg}$$

$$g = 386 \frac{\text{m}}{\text{sec}^2} = 9.81 \text{ m/s}^2$$

$$k = \frac{48EI}{l^3} = \begin{cases} E = 200 \times 10^9 \text{ N/m}^2 \\ l = 2 \times .3048 = .6096 \text{ m} \\ d = .5 \times 2.54 \times 10^{-2} = 1.270 \times 10^{-2} \text{ m} \\ I = \frac{\pi d^4}{64} = .1277 \times 10^{-8} \end{cases}$$

$$k = \frac{200 \times 10^9 \times 48 \times .1277 \times 10^{-8}}{(.6096)^3} = 54116 \text{ N/m}$$

$$.486 \text{ m}_{\text{shaft}} = .486 \times \frac{\pi}{4} (1.270 \times 10^{-2})^2 (.6096) \times \rho = .2938$$

$$\rho = 7830 \frac{\text{kg}}{\text{m}^3} = \text{density of steel}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{54116}{4.534 + .2938}} = 16.86 \text{ Hz}$$

3-18

$$dia = 2.54 \text{ cm} \quad I = \frac{\pi d^4}{64} = \frac{\pi \times 41.62}{64 \times 100^4} = 2.043 \times 10^{-8} \text{ m}^4$$

$$k = \frac{48EI}{l^3} = \frac{48(200 \times 10^9) 2.043 \times 10^{-8}}{.4064^3} = 2.922 \times 10^6 \text{ N/m}$$

$$m = 13.6 + .486 \left(\frac{\pi}{4} \times .0254^2 \times .4064 \right) (7830) = 13.6 + .784 = 14.38 \text{ kg}$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{2.922 \times 10^6}{14.38}} = 71.74 \text{ Hz} = 4304 \text{ rpm}$$

$$\frac{\omega}{\omega_n} = \frac{6000}{4304} = 1.394 \quad h = \frac{e \left(\frac{\omega}{\omega_n} \right)^2}{1 - \left(\frac{\omega}{\omega_n} \right)^2} = -2.060 e$$

$$me = .2879 \text{ kg cm (given)} \quad \therefore e = \frac{.2879}{13.6} = .02117 \text{ cm}$$

$$h = -2.060(.02117) = -.04316 \text{ cm}$$

$$F = m(h+e)\omega^2 = 14.38 \left(\frac{-.04316 + .02117}{100} \right) (2\pi \times 100)^2 = 1273 \text{ N}$$

3-18 Cont.

$$\text{for diam} = 1.905 \text{ cm}$$

$$.486 m_{\text{shaft}} = .486 \left(\frac{\pi}{4} \times .01905^2 \times .4064 \right) (7830) = .4408$$

$$m = m_{\text{disk}} + .483 m_{\text{shaft}} = 14.04 \text{ kg.}$$

$$\bar{I} = \frac{\pi}{64} 1.905^4 \times 100^{-4} = .6464 \times 10^{-8}$$

$$k = \frac{48 (200 \times 10^9) .6464 \times 10^{-8}}{.4064^3} = 0.9441 \times 10^6$$

$$f_n = 41.27 \text{ Hz} = 2476 \text{ rpm} \quad \frac{\omega}{\omega_n} = \frac{6000}{2476} = 2.423$$

$$r = \frac{.02117 (2.423)^2}{1 - (2.423)^2} = -.02552 \quad r + e = .00435$$

$$F = 14.042 \left(\frac{.00435}{100} \right) \left(2\pi \times \frac{6000}{60} \right)^2 = 241.1 \text{ N}$$

3-19

$$r = r_0 + \frac{e\omega t}{2} \quad (\text{see Ex. 3.4-1})$$

$$.0508 = 0 + .0212 (2\pi \times 100) \frac{t}{2}$$

$$t = \frac{.0508}{6.6602} = .0075 \text{ sec}$$

3-20

$$m\ddot{x} = -k(x-y) \quad \text{where } x = \text{displ. of } m \text{ measured from static equilb. position of } m \text{ with } y = 0$$

$$\text{Let } y = Y \sin \frac{2\pi V t}{L}$$

$$\text{then } m\ddot{x} + kx = kY \sin \frac{2\pi V t}{L} = kY \sin \omega t$$

$$\text{where } \omega = \frac{2\pi V}{L}$$

$$\text{Sol is } x = X \sin \omega t$$

$$X = \frac{Y}{1 - (\omega/\omega_n)^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

The most unfavorable speed corresponds to $\frac{\omega}{\omega_n} = 1$

$$\therefore V = \frac{L}{2\pi} \sqrt{\frac{k}{m}}$$

3-21 (Ref. Prob 3-20)

$$\text{Eq. (2.2-9)} \quad f_m = \frac{15.76}{\sqrt{\Delta_{mm}}} = \frac{15.76}{\sqrt{101.6}} = 1.563 \text{ Hz}$$

$$\omega_m = 2\pi f_m = 9.824 \text{ 1/s}$$

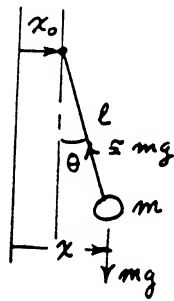
$$V_{\text{crit.}} = \frac{L \omega_m}{2\pi} = \frac{14.63}{2\pi} \times 9.82 = 22.87 \text{ m/s}$$

$$V = 64.4 \frac{\text{km}}{\text{hr}} = \frac{64400}{60^2} = 17.89 \text{ m/s}$$

$$\omega = \frac{2\pi V}{L} = \frac{2\pi \cdot 17.89}{14.63} = 7.683 \text{ 1/s}$$

$$\left(\frac{\omega}{\omega_m}\right)^2 = \left(\frac{7.683}{9.824}\right)^2 = 0.6117 \quad X = \frac{7.62 \text{ cm}}{1 - 0.6117} = 19.62 \text{ cm}$$

3-22



$$x \approx x_0 + l\theta$$

$$\theta = \frac{x - x_0}{l}$$

$$m\ddot{x} = -mg\theta = -\frac{mg}{l}(x - x_0)$$

$$\ddot{x} + \frac{g}{l}x = \frac{g}{l}x_0$$

$$\text{Let } x_0 = X_0 \sin \omega t$$

$$x = X \sin \omega t$$

$$\therefore X = \frac{X_0}{1 - (\omega/\omega_m)^2} \quad \omega_m = \sqrt{\frac{g}{l}}$$

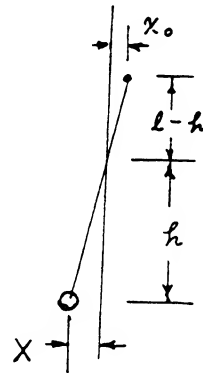
$$\text{when } \omega = \sqrt{2} \omega_m \quad X = -X_0$$

$$\therefore \text{node found at } \frac{l}{2}$$

$$\frac{h}{|X|} = \frac{l-h}{|X_0|}$$

$$h \left| \frac{X_0}{X} \right| = l-h = h \left[\left(\frac{\omega}{\omega_m} \right)^2 - 1 \right] \quad \text{since } \frac{\omega}{\omega_m} > 1$$

$$h = l \left(\frac{\omega}{\omega_m} \right)^2$$



3-23

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$y = Y \sin \omega t$$

$$x = X \sin(\omega t - \phi)$$

$$\dot{x} = \omega X \cos(\omega t - \phi)$$

$$\dot{y} = \omega Y \cos \omega t$$

$$\ddot{x} = -\omega^2 X \sin(\omega t - \phi)$$

Sub. into DE.

$$(k - m\omega^2)X \sin(\omega t - \phi) + c\omega X \cos(\omega t - \phi) = c\omega Y \cos \omega t + kY \sin \omega t$$

Expand $\sin(\omega t - \phi)$ & $\cos(\omega t - \phi)$ & equate coef of $\cos \omega t$ & $\sin \omega t$

$$\text{coef. of } \sin \omega t \rightarrow [(k - m\omega^2)\cos \phi + c\omega \sin \phi]X = kY$$

$$" \quad " \quad \cos \omega t \rightarrow [(k - m\omega^2)\sin \phi - c\omega \cos \phi]X = -c\omega Y$$

Divide & factor $\cos \phi$ from num. & denom. to get

$$\tan \phi = \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2} = \text{Eq. (3.5-9)}$$

Solve for $\sin \phi$ & $\cos \phi$ & sub. into

$$\frac{X}{Y} = \frac{k}{(k - m\omega^2)\cos \phi + c\omega \sin \phi} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \text{Eq. (3.5-8)}$$

after much algebra

3-24

$$f_{\text{cpm}} = 188 \sqrt{\frac{1}{\Delta''} \left(\frac{1}{TR} + 1 \right)} \quad (\text{Sec Sec. 3.6})$$

$$1600_{\text{cpm}} = 188 \sqrt{\frac{1}{\Delta''} \left(\frac{1}{.15} + 1 \right)}, \quad \therefore \Delta'' = .106''$$

25.4 mm = 1 in

$$\therefore \Delta_{\text{mm}} = 2.69 \text{ mm.}$$

For $f = 2200 \text{ cpm}$ the F_{TR} is smaller

3-25

$$TR = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = 0.10 \quad \therefore \left(\frac{\omega}{\omega_n}\right)^2 = 11.0$$

$$\omega_n^2 = \frac{k}{m} = \frac{\omega^2}{11.0} \quad k = \frac{m\omega^2}{11} = \frac{65}{386} \left(\frac{580 \times 2\pi}{60} \right)^2 \frac{1}{11} = 56.5 \text{ lb/in}$$

$$k_{\text{per spring}} = \frac{1}{3} \times 56.5 = 18.8 \text{ lb/in}$$

3-26

$$k = \frac{Mg}{\Delta} = \frac{453.4 \times 9.81}{.005080} = 875561 \frac{N}{m} = 8755.6 \frac{N}{cm}$$

$$\omega_n^2 = \frac{k}{M} = \frac{8755.6 \times 10^2}{453.4} = 1931.1$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{1200 \times 2\pi}{60}\right)^2 \frac{1}{1931.1} = 8.177$$

$$X = \frac{\frac{me}{M} \left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{\frac{.2303}{453.4} \times 8.177}{8.177 - 1} = 0.000579 \text{ m}$$

$$F_{TR} = kX = 875561 \times .000579 = 506.7 \text{ N}$$

3-27

$$\text{New } M = 453.4 + 1136. = 1589.4 \text{ kg}$$

$$\text{New } k = 875561 \times \frac{1589.4}{453.4} = 3069.295 \times 10^3 \frac{N}{m}$$

$\frac{\omega}{\omega_n}$ is same

$$X = \frac{\frac{me}{M} \left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{453.4}{1589.4} \times .000579 = 0.165 \times 10^{-3} \text{ m}$$

3-28

$$M = 68 + 1200 = 1268 \text{ kg.}$$

$$f_n = 160 \text{ cpm} \quad \omega_n = \frac{160}{60} \times 2\pi = 16.75 \text{ r/s}$$

$$\frac{\omega}{\omega_n} = \frac{31.4}{16.75} = 1.8746 \quad k = \omega_n^2 M = 355951 \frac{N}{m}$$

$$X = \frac{100 / 0.3559 \times 10^6}{\sqrt{[1 - 1.875^2]^2 + [0.2 \times 1.875]^2}} = 110.5 \times 10^{-6} \text{ m} \\ = 0.01105 \text{ cm}$$

$$F_{TR} = kX \sqrt{1 + (2\zeta \frac{\omega}{\omega_n})^2} = .3559 \times 10^6 \times 110.5 \times 10^{-6} \sqrt{1 + .1406} \\ = 42.0 \text{ N}$$

3-29

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{280200}{113}} = 49.796$$

$$\omega = 2\pi \times 20 = 125.6 \quad \frac{\omega}{\omega_n} = 2.5236$$

$$\text{Accel.} = 15.24 = \omega^2 Y \quad \therefore Y = .000965 \text{ cm}$$

$$\left| \frac{X}{Y} \right| = \frac{\sqrt{1 + [0.2 \times 2.5236]^2}}{\sqrt{[1 - 6.368]^2 + .2547}} = 0.2078$$

$$\therefore X = .2078 \times .000965 = 0.0002005 \text{ cm}$$

$$\omega^2 X = 3.166 \text{ cm/s}^2 = \text{transmitted accel.}$$

3-30 Add mass M to instrument \therefore increase ω/ω_n

$$X \text{ must be reduced to } \left(\frac{2.03}{3.166} \right) .0002005 = .0001285$$

$$\frac{X}{Y} = \frac{1285}{9650} = .1332 \leq \frac{\sqrt{1 + (.2 \frac{\omega}{\omega_n})^2}}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + (.2 \frac{\omega}{\omega_n})^2}}$$

$$\text{Solve by trial: for } \frac{\omega}{\omega_n} = 3, \quad \sqrt{\quad} = .1321$$

$$\text{for } \frac{\omega}{\omega_n} = 4 \quad \sqrt{\quad} = .0893$$

both of these are OK

$$\text{For } \frac{\omega}{\omega_n} = 4 \quad \omega_n = 31.4 = \sqrt{\frac{k}{M_i + M_o}} = \sqrt{\frac{280200}{M_i + M_o}}$$

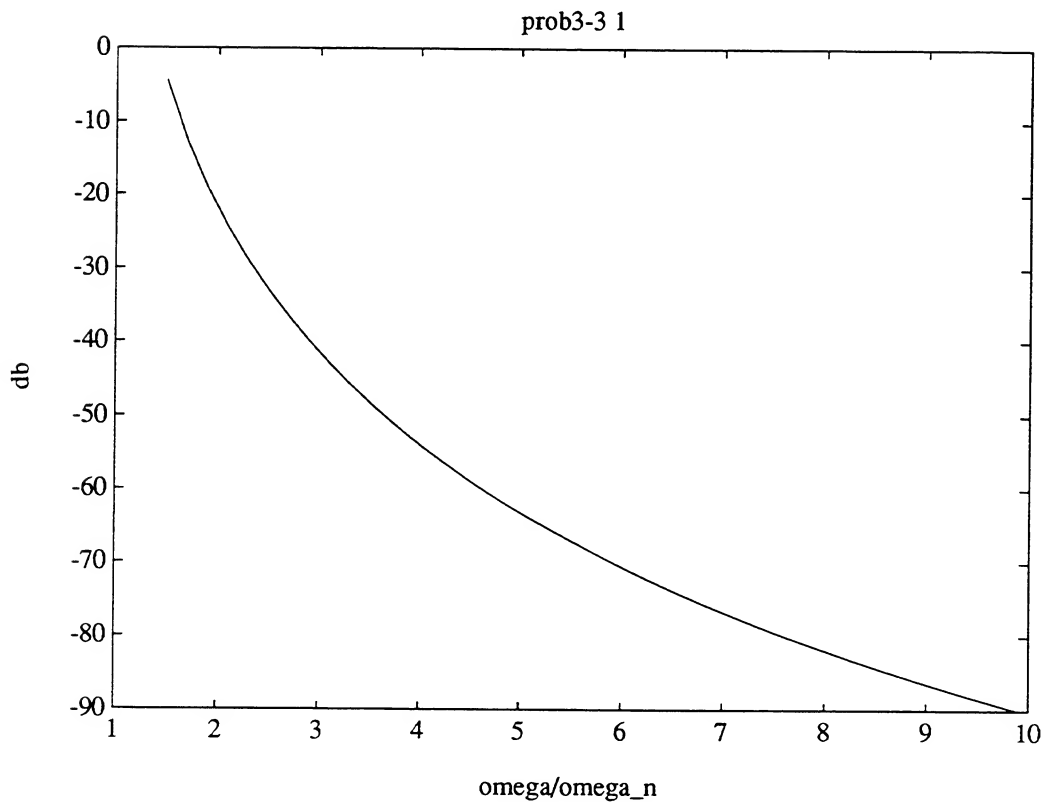
$$M_i + M_o = 284. \quad \therefore M_o = 171 \text{ kg to be added.}$$

3-31 Compare Eq.(3.5-8) with Eq(3.6-2). They are
same

$$TR = \sqrt{\frac{1 + (2\zeta \frac{\omega}{\omega_n})^2}{[1 - (\frac{\omega}{\omega_n})^2]^2 + [2\zeta \frac{\omega}{\omega_n}]^2}}$$

To calculate $20 \log |TR|$, first calculate TR with ζ fixed, varying $\frac{\omega}{\omega_n}$. Then find Db .

```
%This is the code for prob. 3-31
ome=1.5:.2:10;
zeta=.02;
top=1+(2*zeta.*ome).^2;
bot=(1-ome.^2).^2+(2*zeta.*ome).^2;
tr=sqrt(top./bot)
db=20*log(abs(tr));
```



3-32

$$W_d = \pi c \omega X^2 = \pi 2\zeta \frac{\omega}{\omega_m} k X^2$$

$$= \frac{\pi F_0^2}{k} \frac{2\zeta \left(\frac{\omega}{\omega_m}\right)}{\left[1 - \left(\frac{\omega}{\omega_m}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_m}\right]^2}$$

3-33

$$\eta = \frac{W_d}{2\pi U}, \quad U = \frac{1}{2} k X_{\max}^2, \quad W_d = \pi c \omega X_{\max}^2$$

$$\therefore \eta = \frac{\pi c \omega X^2}{2\pi \frac{1}{2} k X^2} = \frac{c \omega}{k}$$

3-34

$$\ddot{x} + 2\zeta \omega_m \dot{x} + \omega_m^2 x = \frac{F}{m} \sin \omega t$$

$$X_{\text{res.}} = \frac{F}{2\zeta k} \quad \text{From Prob. 3-29} \quad \eta_{\text{res}} = \frac{c \omega}{k} = \frac{c \omega_m}{k} \quad \text{at resonan.}$$

$$= \frac{c}{c_0} \frac{2m \omega_m^2}{k} = 2\zeta$$

$$\therefore \ddot{x} + \eta_{\text{res.}} \omega_m \dot{x} + \omega_m^2 x = \frac{F}{m} \sin \omega t$$

3-35

$$\frac{\gamma_d}{\gamma_m} = \frac{\omega_m}{\omega_d} = \frac{1}{\sqrt{1-\zeta^2}} \quad \therefore \left(\frac{\gamma_m}{\gamma_d}\right)^2 = 1 - \zeta^2$$

$$\left(\frac{\gamma_m}{\gamma_d}\right)^2 + \zeta^2 = 1 \quad \text{:: circle of radius 1}$$

3-36

$$\frac{W_d}{U} = \frac{2\zeta \pi k X^2}{\frac{1}{2} k X^2} = 4\zeta \pi$$

$$\text{but } \delta \cong 2\pi \zeta \quad \therefore \frac{W_d}{U} \cong 2\delta, \quad \zeta = \frac{c}{2m \omega_m}$$

$$\delta = \frac{c \pi}{m \omega_m} = \frac{c \pi \omega_m}{k} = \frac{W_d}{2U}$$

$$\underline{3-37} \quad W_d = f(X, \omega) = \pi c \omega X^2$$

$$\delta = \frac{W_d}{2U} = \frac{\pi c \omega X^2}{k X^2} = \frac{\pi c \omega}{k} \quad \text{for viscous damping}$$

$$\underline{3-38} \quad W_d = c_{eq} \pi \omega X^2 = D \dot{X} \quad \text{for Coulomb}$$

$$\therefore c_{eq} = \frac{4D}{\pi \omega X} \quad \zeta_{eq} = \frac{c_{eq}}{C_c} = \frac{4D}{\pi \omega X 2m \omega_n}$$

$$2\zeta_{eq} \frac{\omega}{\omega_n} = \frac{4D}{\pi \omega X 2m \omega_n} \cdot \frac{2\omega}{\omega_n} = \frac{4D}{\pi k X}$$

3-39

$$X = \frac{F_0/k}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + [2\zeta_{eq} \frac{\omega}{\omega_n}]^2}} \quad 2\zeta_{eq} \frac{\omega}{\omega_n} = \frac{4D}{\pi k X}$$

Square both sides and subst for ζ_{eq} .

$$X^2 \left\{ [1 - (\frac{\omega}{\omega_n})^2]^2 + [\frac{4D}{\pi k X}]^2 \right\} = (F_0/k)^2$$

$$X^2 [1 - (\frac{\omega}{\omega_n})^2]^2 = (F_0/k)^2 - (\frac{4D}{\pi k})^2$$

$$X = \frac{\sqrt{(F_0/k)^2 - (4D/\pi k)^2}}{1 - (\frac{\omega}{\omega_n})^2} \quad \therefore \text{for } X \text{ real} \quad F_0 > \frac{4D}{\pi}$$

$$\underline{3-40} \quad \text{Rewrite eq. as } \frac{kX}{F_0} = \frac{\sqrt{1 - (\frac{4D}{\pi F_0})^2}}{1 - (\frac{\omega}{\omega_n})^2} = f\left(\frac{D}{F_0}, \frac{\omega}{\omega_n}\right)$$

$$\underline{3-41} \quad J \ddot{\theta}_2 + K(\theta_2 - \theta_1) = 0 \quad \omega_n^2 = \frac{K}{J}$$

$$\ddot{\theta}_2 + \omega_n^2 \theta_2 = \omega_n^2 \theta_1 = \omega_n^2 \theta_1 \sin \omega t \quad \text{Let } \theta_2 = \Theta_2 \sin \omega t$$

$$\therefore \Theta_2 = \frac{\omega_n^2 \Theta_1}{\omega_n^2 - \omega^2} \quad \text{and} \quad \theta_2 - \theta_1 = (\Theta_2 - \Theta_1) \sin \omega t \quad \begin{cases} \theta_2 = \text{outer wheel} \\ \theta_1 = \text{shaft.} \end{cases}$$

$$\text{Rel. ampl} = (\theta_2 - \theta_1) = \frac{\omega_n^2 \Theta_1}{\omega_n^2 - \omega^2} - \Theta_1 = \frac{\omega^2}{\omega_n^2 - \omega^2} \Theta_1$$

$$= \frac{(\frac{\omega}{\omega_n})^2}{1 - (\frac{\omega}{\omega_n})^2} \Theta_1$$

$$\text{Rel to fixed ref} \quad \theta_2 = \frac{1}{1 - (\frac{\omega}{\omega_n})^2} \theta_1$$

3-42 Let $n = \omega/\omega_m$

$$\frac{Z}{Y} = \frac{n^2}{\sqrt{(1-n^2)^2 + (2\zeta n)^2}} \quad \left(\frac{Z}{Y}\right)^2 = \frac{(n^2)^2}{(1-n^2)^2 + (2\zeta n)^2}$$

$$\frac{\partial}{\partial n^2} \left(\frac{Z}{Y}\right) = 0 \text{ gives } n_p^2 = \frac{1}{1-2\zeta^2} \text{ for peak ampl.}$$

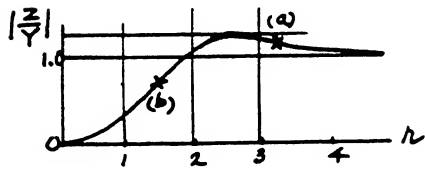
for $\zeta = 0.650$, $n_p = 2.54$, $\frac{Z}{Y} = 1.012$

for (a) $\frac{Z}{Y} = 1.010$ Write above eq. as $n^4[1 - (\frac{Y}{Z})^2] - n^2(2-4\zeta^2) + 1 = 0$

For accuracy expand $1 - (\frac{Y}{Z})^2$

$$= 1 - \frac{1}{(\frac{Z}{Y})^2} = 1 - \frac{1}{(1.01)^2}$$

$$= 1 - (1+0.01)^{-2} = 1 - (1 - 0.02 + 0.0003 - \dots) = 0.0197 \quad (2-4\zeta^2) = 0.310$$



$$0.0197 n^4 - 0.310 n^2 + 1 = 0$$

$$\frac{\omega}{\omega_m} = n = \begin{cases} 2.125 \\ 3.350 \end{cases}$$

\therefore Lowest freq. for 1% accuracy = $3.35 \times 4.75 = \underline{15.9 \text{ cps.}}$

(b) For 2% above plot indicates we must use $\frac{Z}{Y} = 0.98$

$$-0.0412 n^4 - 0.310 n^2 + 1 = 0, \quad n = 1.57$$

$$1.57 \times 4.75 = \underline{7.45 \text{ cps}}$$

3-43

$$Z = \frac{(\frac{\omega}{\omega_m})^2 Y}{1 - (\frac{\omega}{\omega_m})^2}$$

$$Y = Z \left[\left(\frac{\omega_m}{\omega}\right)^2 - 1 \right]$$

$$= 0.052 \left[\left(\frac{1}{4}\right)^2 - 1 \right] = -0.0488 \text{ cm}$$

3-44 sensitivity = 0.096 rms volts/cm per sec.

0.024 rms volts measured \therefore cm per sec vel = $\frac{0.024}{0.096} = 0.250$

$\omega = 2\pi \times 30 = 188.49 \text{ rad/s.}$

(a) vel = 0.250 = ωx = $188.49 x_{\text{cm.}}$ $x = \underline{0.001326 \text{ rms cm.}}$

(b) ampl = 0.40 cm peak

cannot be used since 0.40 cm exceeds clearance of 0.30 cm

3-45 sensitivity = 40 mV/cm per s.

$$1 \text{ g accel} = 9.81 \text{ m/s}^2 = 981 \text{ cm/s}^2 \quad \text{accel} = \omega^2 Y$$

$$(a) \text{ at } 10 \text{ cps} \quad \omega = 2\pi 10 = 62.83 \text{ r/s} \quad \text{vel} = \omega Y.$$

$$\omega Y = \frac{981}{62.83} = 15.61 \text{ cm/s}.$$

$$\text{Output volts} = 40 \times 15.61 = \underline{\underline{624.5 \text{ mV}}}$$

$$(b) \text{ at } 2000 \text{ cps} \quad \omega = 12566$$

$$\omega Y = \frac{981}{12566} = 0.07806 \text{ cm/s}.$$

$$\text{Output volts} = 40 \times 0.07806 = \underline{\underline{3.123 \text{ mV}}}$$

3-46 For vel. pick-up $\frac{\omega}{\omega_n} \gg 1 \quad \therefore Z = Y$

$$\text{Volts generated by instrument} = \omega Z = \omega Y$$

3-47 Sensitivity = 30 mV/cm. per s.

$$3 \text{ mV} = \text{accuracy limit} = 30 (\omega Z) \quad \therefore \omega Z = 0.10 \text{ cm/s} \\ = \text{Limiting vel.}$$

$$v_{\min} = 0.10 = \frac{981}{2\pi f} \quad \therefore f = \underline{\underline{1561 \text{ cps}}} = \text{upper freq. limit.}$$

$$\text{At } f = 200 \quad v = \frac{981}{2\pi \times 200} = 0.7807$$

$$\text{instr. reading} = 30 \times 0.7807 = \underline{\underline{23.42 \text{ mV}}}$$

3-48 $C = 450 \text{ pF} \quad Q = 18 \text{ pC/g}$

$$C_{\text{cable}} = 5 \times 50 = 250 \text{ pF}$$

$$E = \frac{Q}{C} = \frac{18}{450} = 0.040 \text{ volts/g} = \underline{\underline{40 \text{ mV/g}}} \text{ open circuit}$$

With cable

$$E = 40 \times \frac{450}{450 + 250} = \underline{\underline{25.7 \text{ mV/g}}}$$

3-49

$$\frac{W_d}{U} = \frac{\pi c \omega X^2}{\frac{1}{2} k X^2} = \frac{2\pi c \omega}{k}$$

$$c = \zeta c_c = \zeta 2 \sqrt{k m}$$

$$\frac{W_d}{U} = 2\pi \cdot 2\zeta \sqrt{k m} \cdot \frac{\omega}{k} = 4\pi \zeta \frac{\omega}{\omega_m}$$

3-50

$$\begin{aligned} \frac{W_d}{U} &= 4\pi \zeta \frac{\omega}{\omega_m}, & \zeta &\approx \frac{\delta}{2\pi} \\ &= 2\delta \frac{\omega}{\omega_m} \end{aligned}$$

3-51

$$\eta = \frac{W_d}{2\pi U} = \frac{1}{2\pi} (4\pi \zeta \frac{\omega}{\omega_m}) = 2\zeta \frac{\omega}{\omega_m}$$

but $2\zeta = \gamma$ at resonance

$$\therefore \eta = \gamma \frac{\omega}{\omega_m} = \gamma \quad \text{at resonance}$$

3-52

$$x = \frac{(1-r^2)}{(1-r^2)^2 + (2\zeta r)^2}, \quad y = \frac{-2\zeta r}{(1-r^2)^2 + (2\zeta r)^2}$$

$$y + \frac{1}{2(2\zeta r)} = \frac{-2\zeta r}{(1-r^2)^2 + (2\zeta r)^2} + \frac{1}{2(2\zeta r)} = \frac{-2(2\zeta r)^2 + (1-r^2)^2 + (2\zeta r)^2}{2(2\zeta r)[(1-r^2)^2 + (2\zeta r)^2]}$$

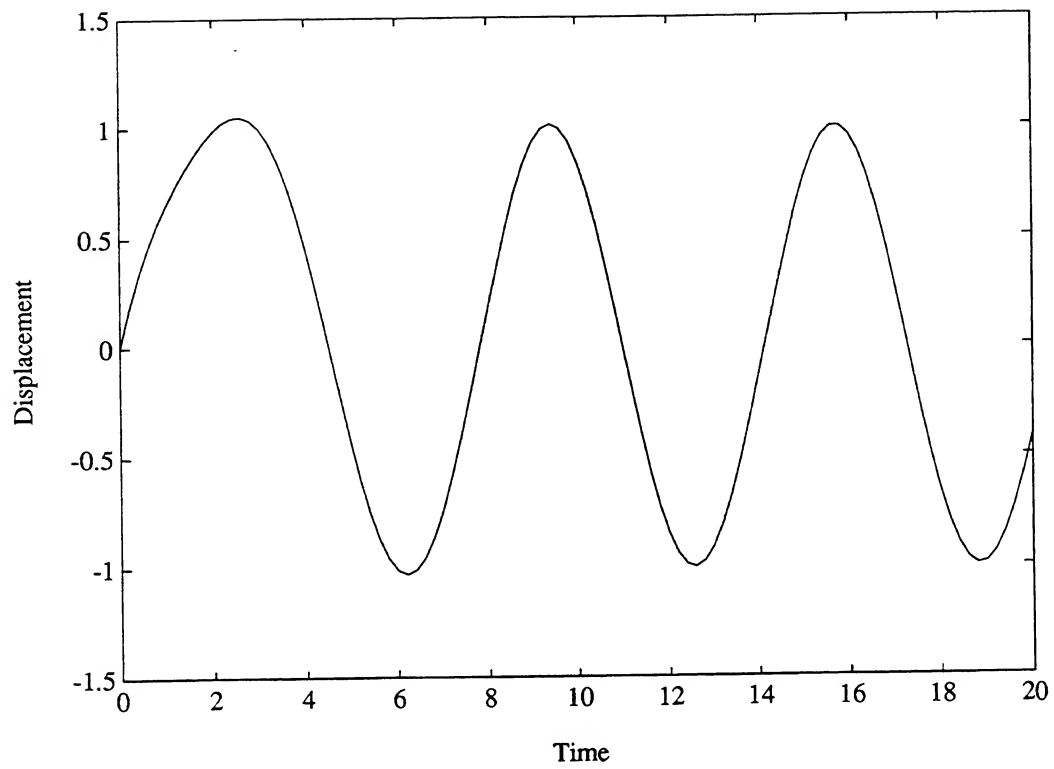
$$= \frac{(1-r^2)^2 - (2\zeta r)^2}{2(2\zeta r)[(1-r^2)^2 + (2\zeta r)^2]}$$

$$\therefore x^2 + \left(y + \frac{1}{2(2\zeta r)}\right)^2 = \frac{(1-r^2)^2}{[(1-r^2)^2 + (2\zeta r)^2]} + \left\{ \frac{(1-r^2)^2 - (2\zeta r)^2}{2(2\zeta r)[(1-r^2)^2 + (2\zeta r)^2]} \right\}$$

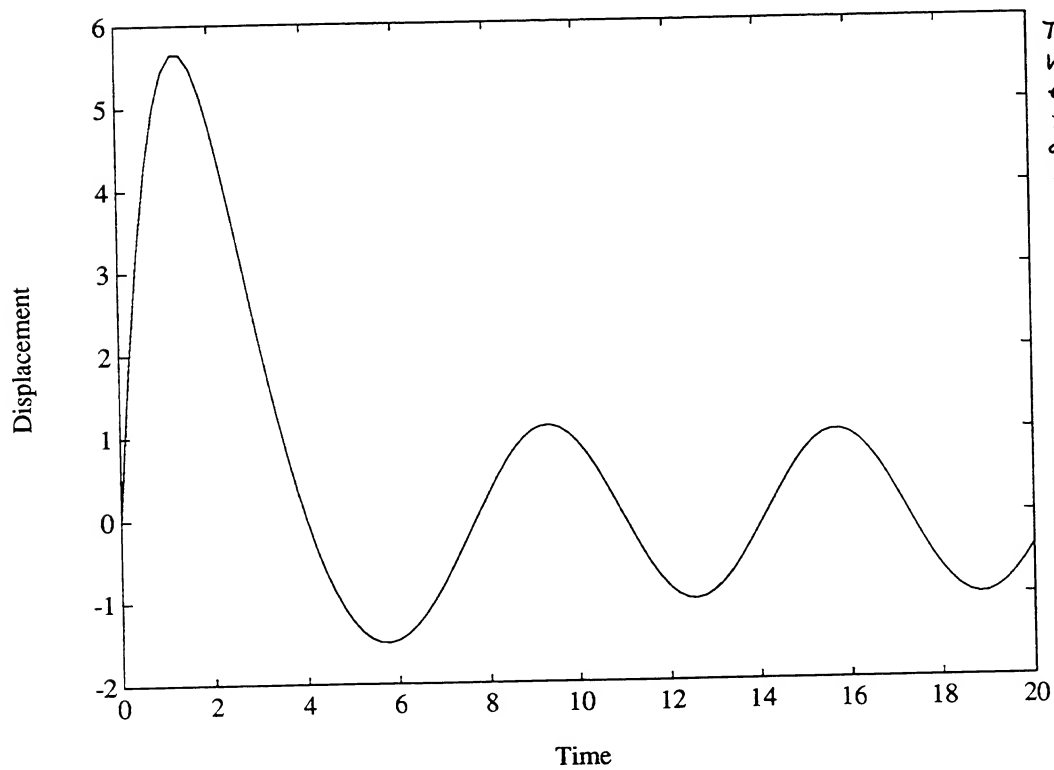
$$= \frac{1}{4(2\zeta r)^2}$$

3-53

3-53 a i



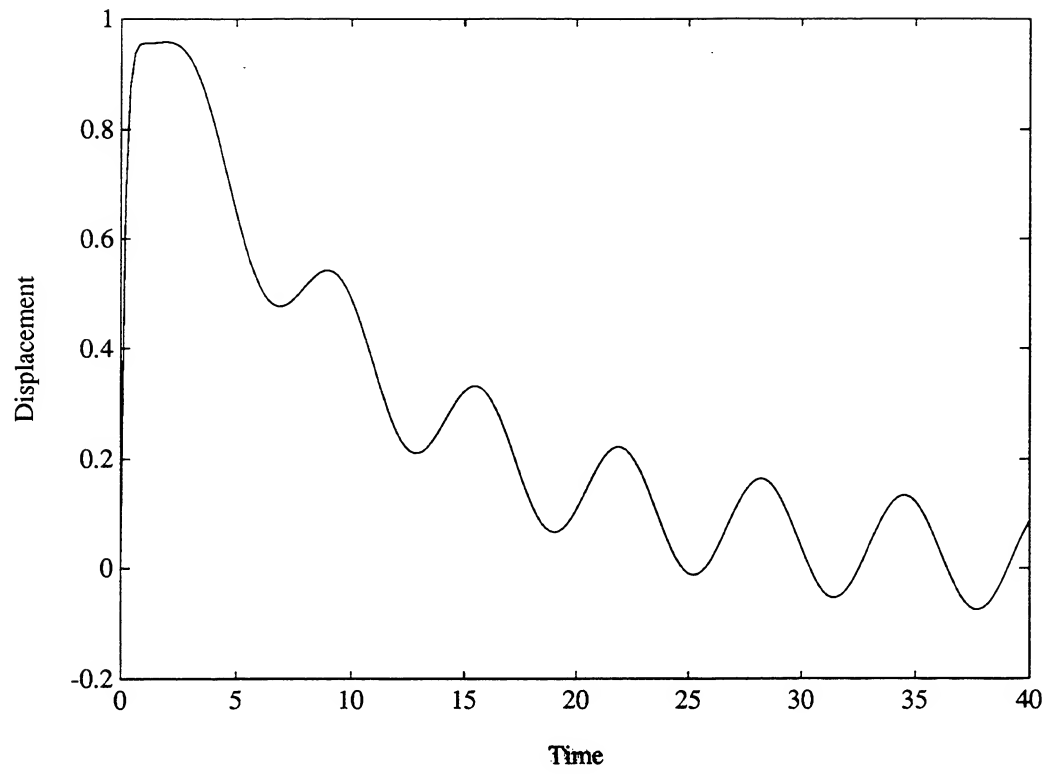
3-53 a ii



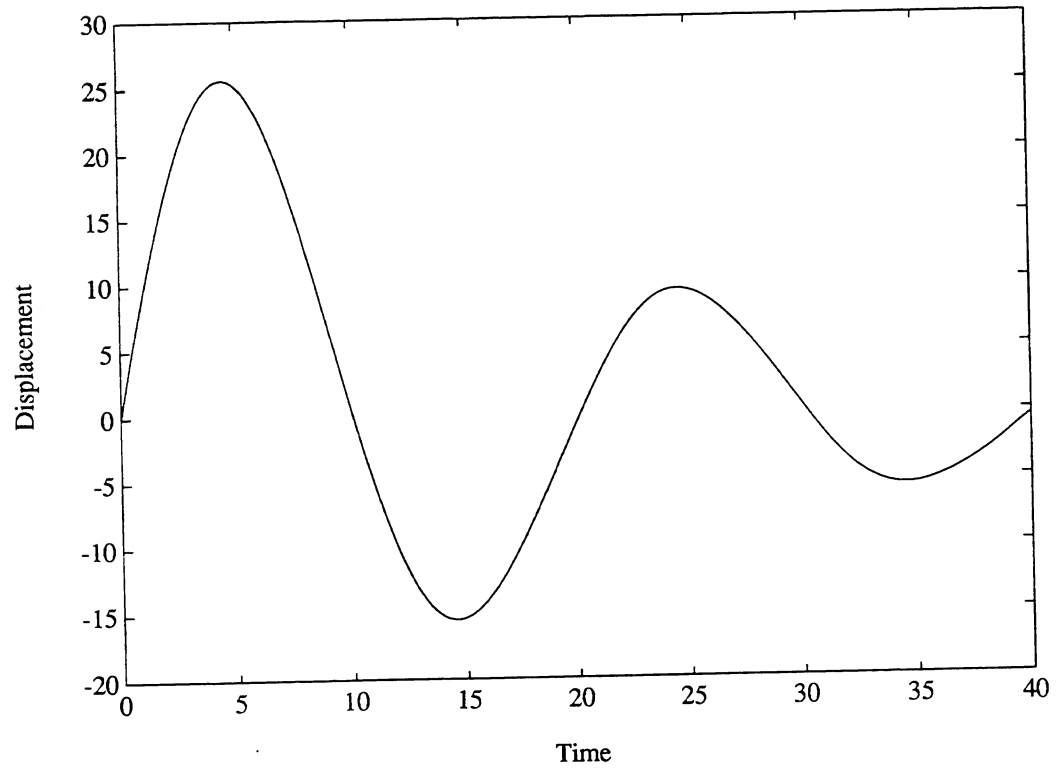
The initial velocity affects the amplitude of the response

3-53 cont

5-20-00

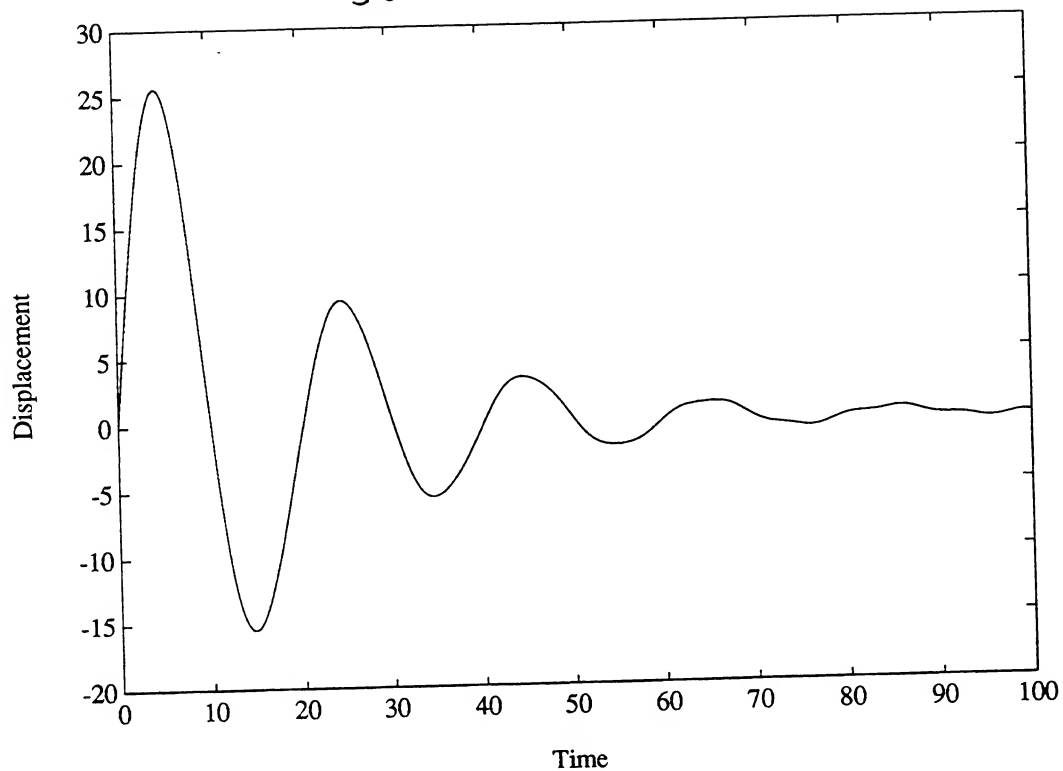


3-53 b ii



3-5-3 cont

3-5-3 b iii



The first system has a larger damping coefficient so it damps out the transient solution faster than the second problem

$$m\ddot{X} + c\dot{X} + kX = F_1 \sin(\omega_1 t) + F_2 \sin(\omega_2 t)$$

Solution of this equation is written as
 $X(t) = X_c(t) + X_p(t)$, where,

$X_c(t)$ is the solution of the homogenous equation, and $X_p(t)$ is the particular solution. Since the differential equation is linear $X_p(t) = X_{p1}(t) + X_{p2}(t)$, where,

$$X_{p_i}(t) = X_i \sin(\omega_i t - \phi_i) \quad \text{with}$$

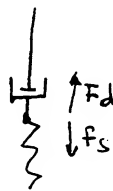
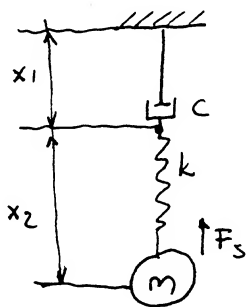
$$\frac{X_i k}{F_i} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_i}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega_i}{\omega_n}\right)\right]^2}} \quad \text{and} \quad \tan \phi_i = \frac{2\zeta\left(\frac{\omega_i}{\omega_n}\right)}{1 - \left(\frac{\omega_i}{\omega_n}\right)^2}$$

for $i=1, 2$.

The plots of $\frac{X_i k}{F_i}$ versus $\frac{\omega_i}{\omega_n}$, $i=1, 2$ are the same the one $\left(\frac{X k}{F_0} \text{ versus } \frac{\omega}{\omega_n}\right)$ in the text

$X_c(t)$ can be the critically damped, overdamped or underdamped depending on the value of ζ . The solution to each of these cases is given in the text.

3-55



$$F_d = -F_s$$

$$k(x_2 - x_{20}) = c\dot{x}_1$$

x_{20} - unstretched length of the spring

$$m(\ddot{x}_1 + \ddot{x}_2) = F_s = k(x_2 - x_{20})$$

$$c\ddot{x}_1 = k\dot{x}_2$$

$$\ddot{x}_1 = \frac{k}{c}\dot{x}_2$$

$$m\left(\frac{k}{c}\dot{x}_2 + \ddot{x}_2\right) = k(x_2 - x_{20})$$

$$\text{Let } z = x_2 - x_{20}$$

$$m\ddot{z} = -m\frac{k}{c}\dot{z} - kz$$

$$c_{\text{eff}} = m\frac{k}{c}$$

Note: c_{eff} inversely proportional to c !
- with bigger c the dashpot becomes stiffer.

The equation of motion

$$m\ddot{z} + m\frac{k}{c}\dot{z} + kz = F_0 \sin \omega t$$

Energy dissipated by cycle:

$$W_d = \oint c_{\text{eff}} \dot{x} dx = \oint c_{\text{eff}} (\dot{x})^2 dt$$

Integration with $\dot{x} = \omega \underline{X} \cos(\omega t - \phi)$, where \underline{X} is the amplitude of motion gives

$$W_d = \pi c_{\text{eff}} \omega \underline{X}^2 = \pi \frac{mk}{c} \omega \underline{X}^2$$

3 - 56

```

clear
global m k a F w
'Problem 3-56 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
k=input('Enter the value of the spring stiffness : ')
a=input('Enter the value of the damping coefficient a : ')
F=input('Enter the amplitude of the forcing function : ')
nc=input('Enter the desired number of cycles: ')
w1=input('Enter the lower bound of the forcing frequency w : ')
w2=input('Enter the upper bound of the forcing frequency w : ')
iw=input('Enter the desired increment of the forcing frequency w : ')
'Wait ..., this might take some time'
t0=0;l=0;
for om=w1:iw:w2
    tf=nc*2*pi/om;
    l=l+1;wf(l)=om;w=om;
    clear t x ampl
    [t,x]=ode45('p350d',t0,tf,x0);
    mx=size(x);n=mx(1,1);
    sum=0;j=0;
    for i=1:n-1
        if (x(i+1,2)<0 | x(i+1,2)==0) & x(i,2)>0
            j=j+1;
            ampl(j)=x(i,1)+(x(i+1,1)-x(i,1))/(x(i+1,2)-x(i,2))*(-x(i,2));
            sum=sum+ampl(j);
        end
    end
    avamp(l)=sum/j;
end
plot(wf,avamp);xlabel('Frequency w');ylabel('Amplitude average')
title('Problem 3-56 solution')

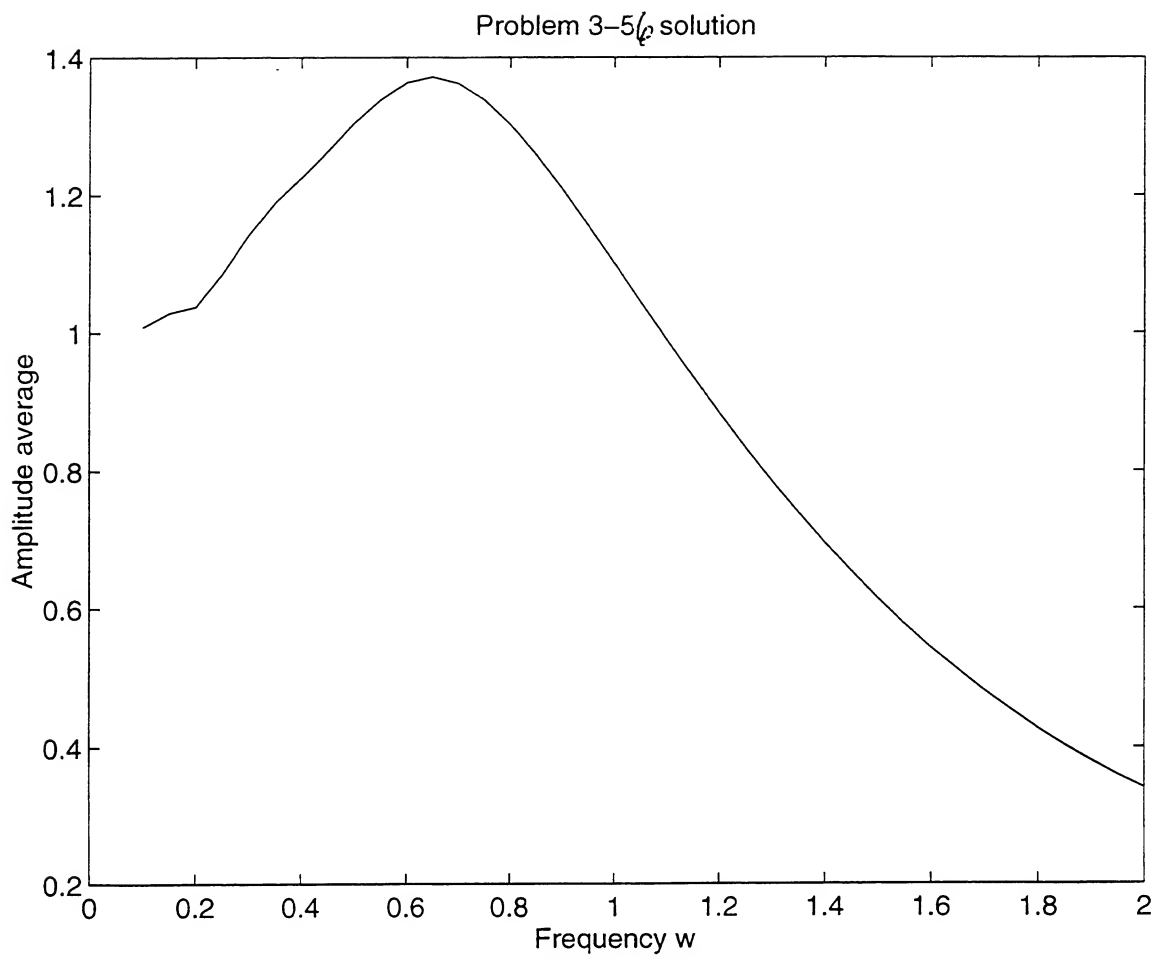
```

```

function xdot=p350d(t,x)
global m k a F w
if x(2)>0 | x(2)==0
    xdot=[x(2);
          (1/m)*(-a*x(2)^2-k*x(1)+F*sin(w*t))];
else
    xdot=[x(2);
          (1/m)*(a*x(2)^2-k*x(1)+F*sin(w*t))];
end

```

3-56 cont.



3-57

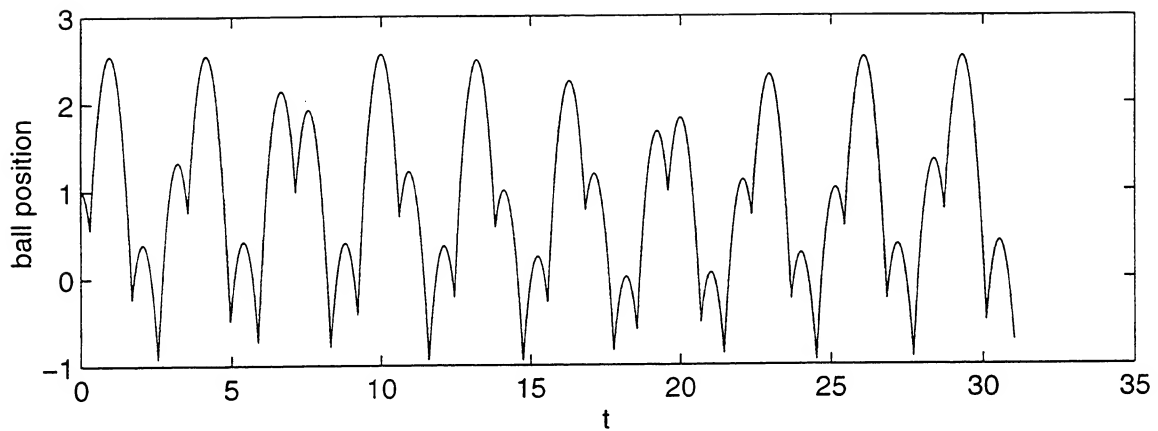
'Problem 3-57 Solution'

```
clear
x0=input('Enter the value of the initial position of the ball >0 : ')
u0=input('Enter the value of the initial velocity of the ball : ')
N=input('Enter the desired number of hits : ')
w=input('Enter the value of the frequency of the table : ')
'Wait...'
inc=(2*pi)/(300*w);
g=9.81;
t0=0;t(1)=0;x(1)=x0;u(1)=u0;y(1)=0;
j=1;
for i=1:N
    while x(j)>y(j) | x(j)==y(j)
        j=j+1;
        t(j)=t(j-1)+inc;
        x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
        y(j)=sin(w*t(j));
        u(j)=u0-g*(t(j)-t0);
    end
    t(j)=t(j-1)+(y(j-1)-x(j-1))*(t(j)-t(j-1))/((x(j)-y(j))-(x(j-1)-y(j-1)));
    thit(i)=t(j);
    x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
    y(j)=sin(w*t(j));
    v=w*cos(w*t(j));
    uneg(i)=u0-g*(t(j)-t0);
    uplus(i)=2*v-uneg(i);u(j)=uplus(i);
    subplot(2,1,2);plot(thit(i),uplus(i),'o');hold on
    u0=uplus(i);t0=t(j);x0=x(j);
end
m=size(x);n=m(1,2);
sum=0;l=0;
for k=2:n-1
    if u(k)<0 & u(k-1)>0
        amp=x(k-1)-u(k-1)*(x(k)-x(k-1))/(u(k)-u(k-1));
        sum=sum+amp; l=l+1;
    elseif u(k+1)<0 & u(k)==0 & u(k-1)>0;
        sum=sum+x(k);l=l+1;
    end
end
end
'The average amplitude of the ball is'
sum/l
subplot(2,1,1);plot(t,x); xlabel('t');ylabel('ball position')
subplot(2,1,2);plot(t,u); xlabel('t');ylabel('ball velocity')
```

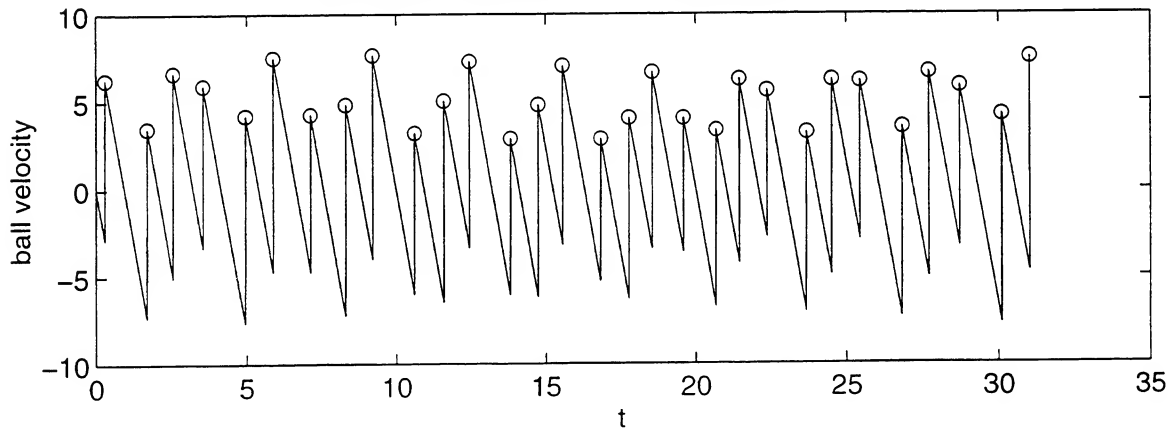
title(' The points o are the points immediately after the hits')

3-57 cont

Initial position = 1
Initial velocity = 0
Frequency $\omega = 2$
 $N = 30$



The points o are the points immediately after the hits



3-57 count

'Problem 3-57 Solution For The Frequency Spectrum'

```
clear
x0=input('Enter the value of the initial position of the ball >0 : ')
u0=input('Enter the value of the initial velocity of the ball : ')
N=input('Enter the desired number of hits : ')
w1=input('Enter the lower bound of the table frequency w : ')
w2=input('Enter the upper bound of the table frequency w : ')
iw=input('Enter the desired increment of the table frequency w : ')
'Wait..., this might take a long time'
x00=x0;u00=u0;
nw=0;
for w=w1:iw:w2
nw=nw+1;wf(nw)=w;
clear t x u y
inc=(2*pi)/(300*w);
g=9.81;
x0=x00;u0=u00;
t0=0;t(1)=0;x(1)=x0;u(1)=u0;y(1)=0;
j=1;
for i=1:N
    while x(j)>y(j) | x(j)==y(j)
        j=j+1;
        t(j)=t(j-1)+inc;
        x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
        y(j)=sin(w*t(j));
        u(j)=u0-g*(t(j)-t0);
    end
    t(j)=t(j-1)+(y(j-1)-x(j-1))*(t(j)-t(j-1))/((x(j)-y(j))-(x(j-1)-y(j-1)));
    thit(i)=t(j);
    x(j)=x0+u0*(t(j)-t0)-0.5*g*(t(j)-t0)^2;
    y(j)=sin(w*t(j));
    v=w*cos(w*t(j));
    uneg(i)=u0-g*(t(j)-t0);
    uplus(i)=2*v-uneg(i);u(j)=uplus(i);
    u0=uplus(i);t0=t(j);x0=x(j);
end
m=size(x);n=m(1,2);
sum=0;l=0;
for k=2:n-1
    if u(k)<0 & u(k-1)>0
        amp=x(k-1)-u(k-1)*(x(k)-x(k-1))/(u(k)-u(k-1));
        sum=sum+amp; l=l+1;
    elseif u(k+1)<0 & u(k)==0 & u(k-1)>0;
```

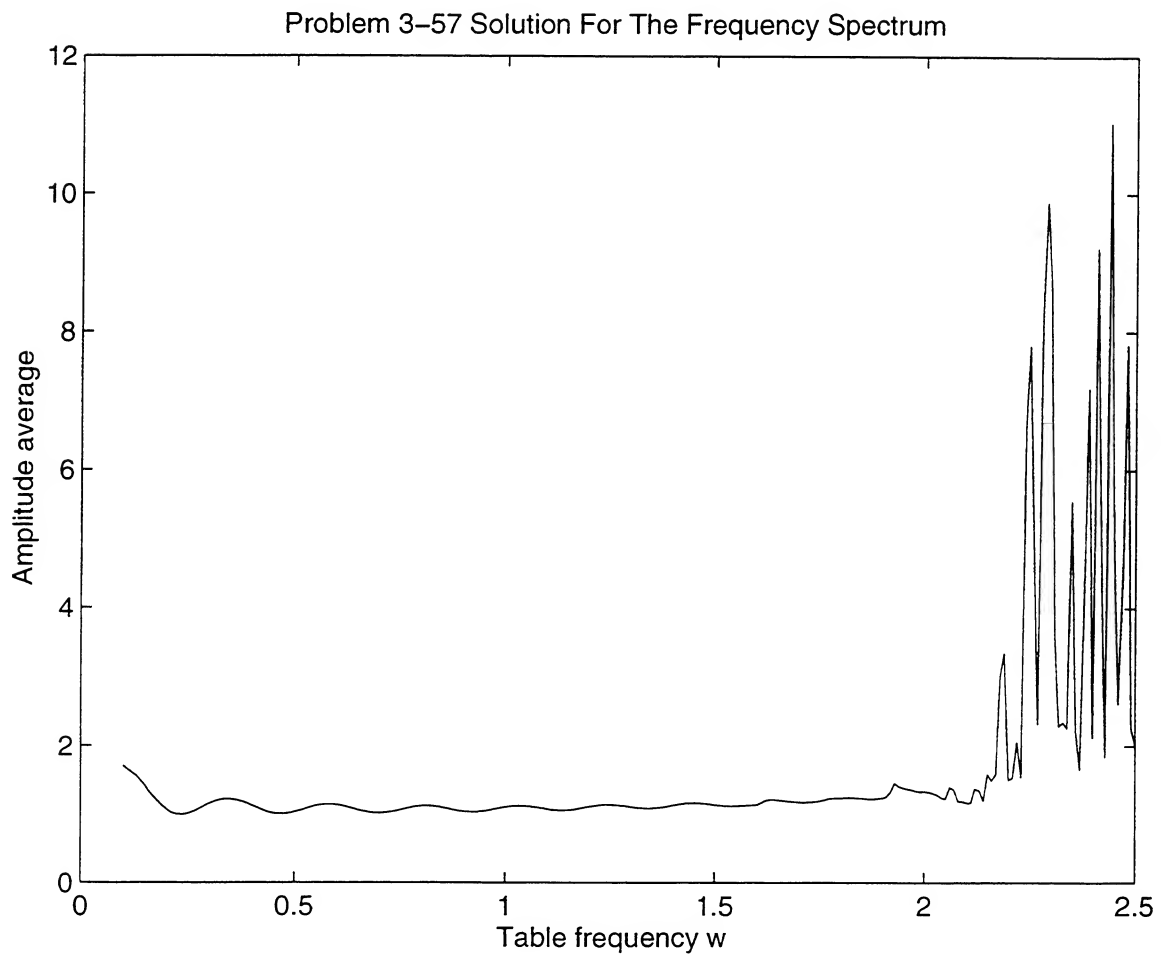
3-57 cont

```

sum=sum+x(k);l=l+1;
end
end
avamp(nw)=sum/l;
end
plot(wf,avamp);xlabel('Table frequency w');ylabel('Amplitude average');
title('Problem 3-57 Solution For The Frequency Spectrum')

```

$x_0 = 1$
 $u_0 = 0$
 $N = 30$
 $w_1 = 0.1$
 $w_2 = 2.5$
 $(w = 0.01)$



4-1

Eq. (4.1-6) for impulsive response

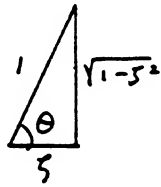
$$x = \frac{\hat{F}}{m\omega_n \sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \sqrt{1-\zeta^2} \omega_n t$$

For max response $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = \frac{\hat{F}}{m\omega_n \sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left\{ -\zeta\omega_n \sin \sqrt{1-\zeta^2} \omega_n t + \sqrt{1-\zeta^2} \omega_n \cos \sqrt{1-\zeta^2} \omega_n t \right\} = 0$$

$$\therefore \tan \sqrt{1-\zeta^2} \omega_n t = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

4-2



Let $\theta = \sqrt{1-\zeta^2} \omega_n t$

$$\tan \theta = \frac{\sqrt{1-\zeta^2}}{\zeta} \quad \text{from Prob 4-1}$$

$$\sin \theta = \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2 + \zeta^2}} = \sqrt{1-\zeta^2}$$

Eq. (4.1-6) becomes $x_{\text{peak}} = \frac{\hat{F}}{m\omega_n \sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}}$

$$\frac{x_{\text{peak}} \sqrt{k/m}}{\hat{F}} = \exp \left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

4-3

From Ex 4.2-1

$$\frac{xk}{F_0} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\sqrt{1-\zeta^2} \omega_n t - \gamma)$$

$$\tan \gamma = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

For peak response $\frac{d}{d(\omega_n t)} \left(\frac{xk}{F_0} \right) = 0$

$$\left\{ -\zeta \cos(\sqrt{1-\zeta^2} \omega_n t - \gamma) - \sqrt{1-\zeta^2} \sin(\sqrt{1-\zeta^2} \omega_n t - \gamma) \right\} = 0$$

$$\therefore \tan(\sqrt{1-\zeta^2} \omega_n t - \gamma) = \frac{-\zeta}{\sqrt{1-\zeta^2}} \quad \text{expand}$$

$$\frac{\tan \sqrt{1-\zeta^2} \omega_n t - \tan \gamma}{1 + \tan \sqrt{1-\zeta^2} \omega_n t \cdot \tan \gamma} = \frac{-\zeta}{\sqrt{1-\zeta^2}}$$

$$\tan \sqrt{1-\zeta^2} \omega_n t - \frac{\zeta}{\sqrt{1-\zeta^2}} = -\frac{\zeta}{\sqrt{1-\zeta^2}} \left[1 + \frac{\zeta}{\sqrt{1-\zeta^2}} \tan \sqrt{1-\zeta^2} \omega_n t \right]$$

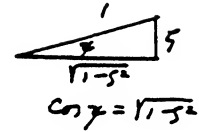
$$\tan \sqrt{1-\zeta^2} \omega_n t \left[1 + \frac{\zeta^2}{1-\zeta^2} \right] = \tan \sqrt{1-\zeta^2} \omega_n t = 0 \quad \therefore \omega_n t = \frac{\pi}{\sqrt{1-\zeta^2}}$$

4-4 Subst. $\omega_n t = \frac{\pi}{\sqrt{1-\zeta^2}}$ into Eq. for peak $\frac{x_k}{F_0}$

$$\left(\frac{x_k}{F_0}\right)_{\text{peak}} = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp\left(\frac{-5\pi}{\sqrt{1-\zeta^2}}\right) \cos(\pi - \gamma)$$

$$= 1 + \frac{1}{\sqrt{1-\zeta^2}} \exp\left(\frac{-5\pi}{\sqrt{1-\zeta^2}}\right) \cos \gamma$$

$$\left(\frac{x_k}{F_0}\right)_{\text{peak}} = 1 + \exp\left(\frac{-5\pi}{\sqrt{1-\zeta^2}}\right)$$



4-5

From Eq. 4.4-4 the response during the first step function is

$$\frac{x}{F_0} = (1 - \cos \omega_n t)$$

At time t_1 , the displacement and velocity are

$$\left(\frac{x}{F_0}\right)_1 = (1 - \cos \omega_n t_1)$$

$$\left(\frac{\dot{x}}{F_0}\right)_1 = \omega_n \sin \omega_n t_1$$

which become the initial conditions for the second phase. Since the excitation is zero during the time $t > t_1$, the above equations are substituted into the general solution for the free vibration

$$x(t) = \frac{\dot{x}(t_1)}{\omega_n} \sin \omega_n t + x(t_1) \cos \omega_n t \quad t > t_1$$

$$x(t) = \frac{F_0}{k} \left\{ \sin \omega_n t_1 \sin \omega_n t - (1 - \cos \omega_n t_1) \cos \omega_n t \right\} \quad t > t_1$$

$$\frac{x(t)}{F_0} = \left\{ -\cos \omega_n t + \cos \omega_n t \cos \omega_n t_1 + \sin \omega_n t \sin \omega_n t_1 \right\}$$

$$= \left\{ -\cos \omega_n t + \cos \omega_n (t - t_1) \right\} \quad t > t_1$$

4-6

With zero initial conditions Eq. 4.2-1 gives for the particular integral, the equation

$$x(t) = \int_0^t f(\xi) \sin \omega_n(t-\xi) d\xi$$

where $f(\xi)$ is the arbitrary excitation force.

The solution for the homogeneous equation is found from Eq. 2.2-6

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t$$

and the complete solution is the sum of these equations.

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t + \int_0^t f(\xi) \sin \omega_n(t-\xi) d\xi$$

4-7

From Eq. 4.2-1, the response to a unit step function is

$$g(t) = \int_0^t h(t-\xi) d\xi$$

Differentiate both sides w.r.t. time

$$\dot{g}(t) = h(t)$$

4-8 Eq. 4.2-1 $x = \int_0^t f(\xi) h(t-\xi) d\xi$

From Prob. 4-7 $\dot{g}(t-\xi) = h(t-\xi)$

$x(t) = \int_0^t f(\xi) \dot{g}(t-\xi) d\xi$ Integrate by parts

$$\begin{aligned} x(t) &= -f(\xi)g(t-\xi) \Big|_0^t + \int_0^t \dot{f}(\xi)g(t-\xi) d\xi \\ &= -f(t)g(0) + f(0)g(t) + \int_0^t \dot{f}(\xi)g(t-\xi) d\xi \quad \text{but } g(0) = 0 \\ \therefore x(t) &= f(0)g(t) + \int_0^t \dot{f}(\xi)g(t-\xi) d\xi \end{aligned}$$

4-9 $\bar{x}(s) = \frac{(ms+c)x(0) + m\dot{x}(0)}{ms^2 + cs + k}$

$$= \frac{(s+2.5\omega_n)x(0)}{s^2 + 2.5\omega_n s + \omega_n^2} + \frac{\dot{x}(0)}{s^2 + 2.5\omega_n s + \omega_n^2}$$

$\mathcal{L}^{-1} \frac{1}{s^2 + 2.5\omega_n s + \omega_n^2} = \frac{1}{\omega_n \sqrt{1-5^2}} e^{-5\omega_n t} \sin \sqrt{1-5^2} \omega_n t$ see Append. B.

$\mathcal{L}^{-1} \frac{s}{s^2 + 2.5\omega_n s + \omega_n^2} = \text{derivative of above} = e^{-5\omega_n t} \left\{ \cos \sqrt{1-5^2} \omega_n t - \frac{5\omega_n}{\omega_n \sqrt{1-5^2}} \sin \sqrt{1-5^2} \omega_n t \right\}$

$\therefore x(t) = e^{-5\omega_n t} \left\{ \frac{\dot{x}(0) + 5\omega_n x(0)}{\omega_n \sqrt{1-5^2}} \sin \sqrt{1-5^2} \omega_n t + x(0) \cos \sqrt{1-5^2} \omega_n t \right\}$

4-10 $\ddot{y}(t) = 20u(t) - 100t \quad \therefore \ddot{y}(t) = 20\delta(t) - 100$

$\ddot{z} + \omega_n^2 z = -\ddot{y} = 100 - 20\delta(t)$

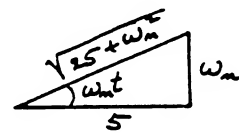
$\bar{z}(s) = \frac{100}{s(s^2 + \omega_n^2)} - \frac{20}{(s^2 + \omega_n^2)}$

$z(t) = \frac{100}{\omega_n^2} (1 - \cos \omega_n t) - \frac{20}{\omega_n} \sin \omega_n t$

$\dot{z}(t) = \frac{100}{\omega_n} \sin \omega_n t - 20 \cos \omega_n t = 0$ for max z

$\therefore \tan \omega_n t = \frac{\omega_n}{5}$ (see Fig)

$z_{\max} = \frac{100}{\omega_n} \left[1 - \frac{5}{\sqrt{25 + \omega_n^2}} \right] - \frac{20}{\omega_n} \frac{\omega_n}{\sqrt{25 + \omega_n^2}} \quad \omega_n = 10$



4-11 $F_0 \sin \omega t = F_0 \sin \frac{\pi}{t_1} t$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \frac{\pi}{t_1} t$$

$$\omega_n = \frac{2\pi}{\tau}$$

Gen Sol. $x(t) = A \sin \omega_n t + B \cos \omega_n t + \frac{F_0/m \sin \frac{\pi}{t_1} t}{1 - (\frac{\pi}{t_1 \omega_n})^2}$

For $x(0) = \dot{x}(0) = 0$, $B = 0$ and $A = \frac{-F_0/m (\frac{\pi}{t_1 \omega_n})}{1 - (\frac{\pi}{t_1 \omega_n})^2}$

With $\frac{\pi}{t_1 \omega_n} = \frac{\pi}{t_1} \frac{\tau}{2\pi} = \frac{\tau}{2t_1}$

$$x(t) = \frac{F_0}{m} \left\{ \frac{-\frac{\tau}{2t_1} \sin \frac{2\pi t}{\tau}}{1 - (\frac{\tau}{2t_1})^2} + \frac{\sin \frac{\pi t}{t_1}}{1 - (\frac{\tau}{2t_1})^2} \right\}$$

$$= \frac{F_0/m}{(\frac{\tau}{2t_1} - \frac{2t_1}{\tau})} \left\{ \sin \frac{2\pi t}{\tau} - \frac{2t_1}{\tau} \sin \frac{\pi t}{t_1} \right\} \quad t < t_1$$

For $t > t_1$, add same solution with t replaced by $(t - t_1)$

4-12 The triangular force can be represented by

$$F_1 = \frac{2F_0}{t_1} t \quad 0 < t < t_1/2$$

$$F_2 = -\frac{4F_0}{t_1} (t - t_1/2) + F_1 \quad t_1/2 < t < t_1$$

$$F_3 = \frac{2F_0}{t_1} (t - t_1) + F_2 \quad t_1 < t$$

Diff. Eq. for $0 < t < t_1/2$ is

$$\ddot{x} + \omega_n^2 x = \frac{2F_0}{m t_1} t = \left(\frac{2\omega_n^2 F_0}{k t_1} \right) t = C t$$

$$\bar{x}(s) = \frac{C}{s^2(s^2 + \omega_n^2)} \quad \therefore x_1(t) = \frac{C}{\omega_n^3} (\omega_n t - \sin \omega_n t) = \frac{2F_0}{k} \left(\frac{t}{t_1} - \frac{\tau}{2\pi t_1} \sin \frac{2\pi t}{\tau} \right)$$

For F_2 additional excitation is $-2Ct + Ct_1 = -2C(t - t_1/2)$

Thus $x_1(t)$ must be supplemented by $-2x_1(t - t_1/2)$

$$x_2(t) = x_1(t) - 2x_1(t - \frac{1}{2}t_1) \quad \frac{1}{2}t_1 < t < t_1$$

For F_3 the excitation in addition to F_2 is $C(t - t_1)$

$$x_3(t) = x_2(t) + x_1(t - t_1) \quad t_1 < t$$

4-11 $F_0 \sin \omega t = F_0 \sin \frac{\pi}{t_1} t$

$$\ddot{x} + \omega_m^2 x = \frac{F_0}{m} \sin \frac{\pi}{t_1} t \quad \omega_m = \frac{2\pi}{T}$$

Gen Sol. $x(t) = A \sin \omega_m t + B \cos \omega_m t + \frac{F_0/m \sin \frac{\pi}{t_1} t}{1 - (\frac{\pi}{t_1 \omega_m})^2}$

For $x(0) = \dot{x}(0) = 0$, $B = 0$ and $A = \frac{-F_0/m (\frac{\pi}{t_1 \omega_m})}{1 - (\frac{\pi}{t_1 \omega_m})^2}$

With $\frac{\pi}{t_1 \omega_m} = \frac{\pi}{t_1} \frac{T}{2\pi} = \frac{T}{2t_1}$

$$x(t) = \frac{F_0}{m} \left\{ \frac{-\frac{T}{2t_1} \sin \frac{2\pi t}{T}}{1 - (\frac{T}{2t_1})^2} + \frac{\sin \frac{\pi t}{t_1}}{1 - (\frac{T}{2t_1})^2} \right\}$$

$$= \frac{F_0/m}{(\frac{T}{2t_1} - \frac{2t_1}{T})} \left\{ \sin \frac{2\pi t}{T} - \frac{2t_1}{T} \sin \frac{\pi t}{t_1} \right\} \quad t < t_1$$

For $t > t_1$, add same solution with t replaced by $(t - t_1)$

4-12 The triangular force can be represented by

$$F_1 = \frac{2F_0}{t_1} t \quad 0 < t < t_{1/2}$$

$$F_2 = -\frac{4F_0}{t_1} (t - t_{1/2}) + F_1 \quad t_{1/2} < t < t_1$$

$$F_3 = \frac{2F_0}{t_1} (t - t_1) + F_2 \quad t_1 < t$$

Diff. Eq. for $0 < t < t_{1/2}$ is

$$\ddot{x} + \omega_m^2 x = \frac{2F_0}{m t_1} t = \left(\frac{2\omega_m^2 F_0}{k t_1} \right) t = Ct$$

$$\bar{x}(s) = \frac{C}{s^2(s^2 + \omega_m^2)} \quad \therefore x_1(t) = \frac{C}{\omega_m^3} (\omega_m t - \sin \omega_m t) = \frac{2F_0}{k} \left(\frac{t}{t_1} - \frac{T}{2\pi t_1} \sin \frac{2\pi t}{T} \right)$$

For F_2 additional excitation is $-2Ct + Ct_1 = -2C(t - t_{1/2})$

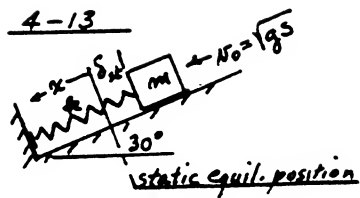
Thus $x_1(t)$ must be supplemented by $-2x_1(t - t_{1/2})$

$$x_2(t) = x_1(t) - 2x_1(t - \frac{1}{2}t_1) \quad \frac{1}{2}t_1 < t < t_1$$

For F_3 the excitation in addition to F_2 is $C(t - t_1)$

$$x_3(t) = x_2(t) + x_1(t - t_1) \quad t_1 < t$$

4-13



$$x = x(0) \cos \omega_n t + \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t$$

$$\dot{x} = -\omega_n x(0) \sin \omega_n t + \dot{x}(0) \cos \omega_n t$$

$$k \delta_{st} = \frac{1}{2} mg$$

When returned to initial position

$$\begin{cases} -\delta_{st} = -\delta_{st} \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t \\ -v_0 = \omega_n \delta_{st} \sin \omega_n t + v_0 \cos \omega_n t \end{cases} \quad \begin{cases} \div \text{ by } \cos \omega_n t \text{ and} \\ \div \text{ the two eqs.} \end{cases}$$

$$\frac{\delta_{st}}{v_0} = \frac{-\delta_{st} + \frac{v_0}{\omega_n} \tan \omega_n t}{\omega_n \delta_{st} \tan \omega_n t + v_0}, \quad \tan \omega_n t = \frac{2 \delta_{st} \omega_n v_0}{v_0^2 - \delta_{st}^2 \omega_n^2}$$

$$\tan \omega_n t = \frac{\sqrt{\frac{mg}{k}}}{5 - \frac{g \cdot m}{4k}}$$

$$\omega_n^2 = k/m$$

$$\delta_{st} = \frac{g}{\omega_n^2}$$

$$v_0 = \sqrt{g \cdot 5}$$

4-14

$$\Delta = \frac{38.6}{6.40} = 6.04", \quad \text{Max displ.} = 2\Delta = 12.08"$$

$$\omega_n = \sqrt{\frac{6.40 \times 386}{88.6}} = 8.0 \text{ rad/s.} = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{8} = 0.784 \text{ s.}$$

$$t_{\text{max displ}} = \frac{T}{2} = 0.392 \text{ s.}$$

4-15

$$m \ddot{x} = -k(x-y) - f$$

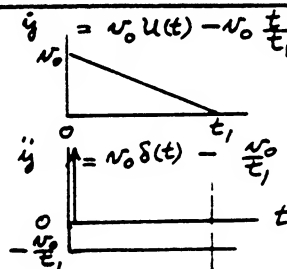
$$\text{Let } z = x - y$$

$$\ddot{z} + \omega_n^2 z = -\ddot{y} - \frac{f}{m}$$

Gen Sol. from Eq. (4.2-5) is

$$\begin{aligned} z &= \frac{-1}{\omega_n} \int_0^t [\ddot{y}(\xi) + \frac{f}{m}] \sin \omega_n(t-\xi) d\xi \\ &= -\frac{v_0}{\omega_n} \int_0^t \left[\delta(t) - \frac{1}{t_1} + \frac{f}{v_0 m} \right] \sin \omega_n(t-\xi) d\xi \end{aligned}$$

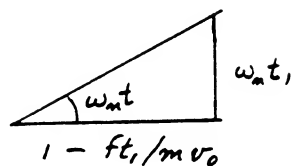
$$= \frac{v_0}{\omega_n} \left\{ \frac{1}{\omega_n t_1} \left(1 - \frac{f t_1}{m v_0} \right) (1 - \cos \omega_n t) - \sin \omega_n t \right\}$$



4-16 Differentiate Z in Prob. 4-15

$$\frac{dz}{dt} = \frac{v_0}{\omega_m} \left\{ \frac{1}{t_1} \left(1 - \frac{ft_1}{mv_0} \right) \sin \omega_m t - \omega_m \cos \omega_m t \right\} = 0$$

$$\therefore \tan \omega_m t = \frac{\omega_m t_1}{1 - \frac{ft_1}{mv_0}}$$



$$\sin \omega_m t = \frac{\omega_m t_1}{\sqrt{(\omega_m t_1)^2 + [1 - ft_1/mv_0]^2}}$$

$$\cos \omega_m t = \frac{(1 - ft_1/mv_0)}{\sqrt{\text{same}}}$$

subst into eq. for Z

$$\begin{aligned} \frac{\omega_m Z_{\max}}{v_0} &= \frac{1}{\omega_m t_1} \left(1 - \frac{ft_1}{mv_0} \right) \left\{ 1 - \frac{(1 - \frac{ft_1}{mv_0})}{\sqrt{(\omega_m t_1)^2 + [1 - ft_1/mv_0]^2}} \right\} - \frac{\omega_m t_1}{\sqrt{(\omega_m t_1)^2 + [1 - ft_1/mv_0]^2}} \\ &= \frac{1}{\omega_m t_1} \left(1 - \frac{ft_1}{mv_0} \right) \left\{ 1 - \frac{\frac{1}{\omega_m t_1} (1 - \frac{ft_1}{mv_0})}{\sqrt{1 + [\frac{1}{\omega_m t_1} (1 - \frac{ft_1}{mv_0})]^2}} \right\} - \frac{1}{\sqrt{1 + [\frac{1}{\omega_m t_1} (1 - \frac{ft_1}{mv_0})]^2}} \end{aligned}$$

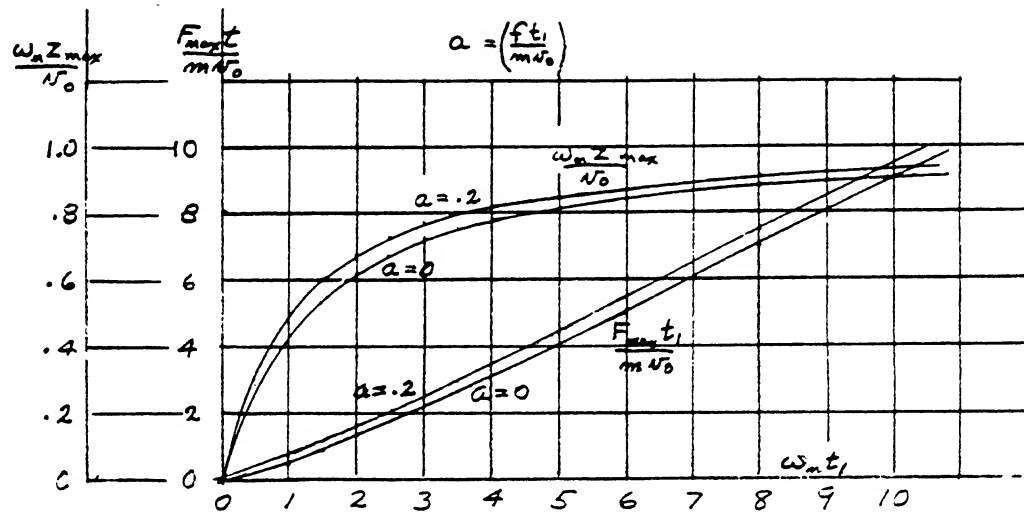
4-17 Let $a = \frac{ft_1}{mv_0}$ and $x = \omega_m t$ $y = \left| \frac{\omega_m Z_{\max}}{v_0} \right|$

$$\left| \frac{\omega_m Z_{\max}}{v_0} \right| = \left| \frac{1}{x} (1-a) \left\{ 1 - \frac{(1-a)}{\sqrt{x^2 + (1-a)^2}} \right\} - \frac{x}{\sqrt{x^2 + (1-a)^2}} \right|$$

$$\left| \frac{F_{\max} t_1}{mv_0} \right| = a + \left| (\omega_m t_1)^2 \left(\frac{\omega_m Z_{\max}}{v_0} \right) \frac{1}{\omega_m t_1} \right| = a + x \left| \left(\frac{\omega_m Z_{\max}}{v_0} \right) \right|$$

x	$y_{a=0}$	$\left \frac{F_{\max} t_1}{mv_0} \right _{a=0}$	$y_{a=.20}$	$\left \frac{F_{\max} t_1}{mv_0} \right _{a=.2}$
1	.4142	.4142	.4806	.6806
2	.6180	1.2360	.6770	1.554
3	.7208	2.1624	.7683	2.504
4	.7808	3.1232	.8198	3.479
5	.8198	4.0990	.8527	4.466
6	.8471	5.0826	.8755	5.453
8	.8828	7.0624	.9050	7.440
10	.9050	9.050	.9232	9.432

4-17 Cont.:



4-18 The response to the ramp function of FIG 4.4-2 is given by Eq. 4.4-3 as

$$x(t) = \frac{F_0}{k} \left[1 - \frac{\sin \omega_n t}{\omega_n t_1} + \frac{1}{\omega_n t_1} \sin \omega_n (t - t_1) \right] \quad t > t_1$$

Differentiating this equation w.r.t. and equating it to zero, the peak time t_p is obtained as

$$\tan \omega_n t_p = \frac{1 - \cos \omega_n t_1}{\sin \omega_n t_1}$$

Since $\omega_n t_p$ must be greater than π , we also obtain

$$\sin \omega_n t_p = -\sqrt{\frac{1}{2}(1 - \cos \omega_n t_1)}$$

$$\cos \omega_n t_p = \frac{-\sin \omega_n t_1}{\sqrt{2(1 - \cos \omega_n t_1)}}$$

Substituting these quantities into $x(t)$, the peak amplitude is found as

$$\left(\frac{xk}{F_0} \right)_{\max} = 1 + \frac{1}{\omega_n t_1} \sqrt{2(1 - \cos \omega_n t_1)}$$

Letting $\gamma = \frac{2\pi}{\omega_n}$ be the period of the oscillator, the above equation is plotted against t_1/γ as in FIG P4-18.

4-19 For small $\frac{t_1}{\gamma}$ the sine pulse approaches an impulse

$$\hat{F} = F_0 \int_0^{t_1} \sin \frac{\pi t}{t_1} dt = \frac{2}{\pi} F_0 t_1$$

$$\text{Response} = x = \frac{\hat{F}}{m \omega_n} \sin \omega_n t = \frac{F_0}{k} \frac{4 t_1}{\gamma} \sin \frac{2\pi t}{\gamma}$$

$$\left(\frac{xk}{F_0} \right)_{\max} = \frac{4 t_1}{\gamma} \quad \text{at } t_p = \gamma/4 \quad \text{Peak response must occur at time } \gg t_1 \quad \therefore \frac{t_1}{\gamma} \ll \frac{1}{4}$$

$$\text{Let } \alpha = \frac{t_1}{\gamma}, \quad \xi = \frac{t}{t_1} \quad \text{for } t \geq t_1$$

$$\left(\frac{xk}{F_0} \right) = \frac{1}{\left(\frac{1}{2\alpha} - 2\alpha \right)} \left[\sin 2\pi\alpha\xi + \sin 2\pi\alpha(\xi-1) \right]$$

When $\alpha = \frac{1}{2}$ above eq. is indeterminate = $\frac{0}{0}$

\therefore differentiate num. & den, w.r.t. α & divide

$$\left(\frac{xk}{F_0} \right)_{\alpha=\frac{1}{2}} = -\frac{\pi}{2} \cos \pi\xi \quad \text{which is max when } \xi=1 = \frac{t}{t_1}, \quad \therefore t_p = t_1$$

$$\text{and } \left(\frac{xk}{F_0} \right)_{\alpha=\frac{1}{2}, \xi=1} = \frac{\pi}{2} = \underline{\underline{1.57}}$$

$$\underline{4-20} \quad m = \frac{16.1}{386} = 0.0417 \quad \frac{t_1}{\gamma} = \frac{0.40}{0.50} = 0.80$$

$$\text{From Fig P4-21} \quad \left(\frac{xk}{F_0} \right)_{\max} = 1.54$$

$$\omega_n = \frac{2\pi}{0.50} = 4\pi \quad k = m \omega_n^2 = 0.0417 (4\pi)^2 = 6.585 \text{ #/in}$$

$$x_{\max} = 1.54 \frac{F}{k} \quad \hat{F} = 2.0 \text{ lbs.} = \frac{1}{2} \times 4.0 \times F_0$$

$$\therefore F_0 = 10$$

$$x_{\max} = 1.54 \times \frac{10}{6.585} = \underline{\underline{2.339''}}$$

4-21 Differentiate 3rd eq. Prob. 4-12

$$\frac{dx}{dt} = \frac{2F_0}{k t_1} \left\{ 2 \cos \frac{2\pi t}{T} \left(\frac{t}{t_1} - 0.5 \right) - \cos \frac{2\pi t}{T} \left(\frac{t}{t_1} - 1.0 \right) - \cos \frac{2\pi t}{T} \right\} = 0$$

$$\text{or } 2 \cos \frac{2\pi t}{T} \left(\frac{t_p}{t_1} - 0.50 \right) - \cos \frac{2\pi t}{T} \left(\frac{t_p}{t_1} - 1 \right) - \cos \frac{2\pi t_p}{T} = 0$$

which gives $t_p = t_1$

4-22 With suggested subst for $A \cos \phi$ and $A \sin \phi$

$$x = \frac{A}{k} \{ \sin \omega_n t \cos \phi - \cos \omega_n t \sin \phi \} = \frac{A}{k} \sin(\omega_n t - \phi)$$

Max response occurs when $(\omega_n t - \phi) = \frac{\pi}{2}$

$$x_{\max} = \frac{A}{k} \quad \text{where} \quad A = \omega_n \sqrt{\left[\int_0^t f(\xi) \sin \omega_n \xi d\xi \right]^2 + \left[\int_0^t f(\xi) \cos \omega_n \xi d\xi \right]^2}$$

4-23 Let $\frac{\pi}{t_1} = p$, $r = \frac{p}{\omega_n} = \frac{\pi}{t_1} \frac{T}{2\pi} = \frac{T}{2t_1}$

and rewrite Eq. 4.4-(C)4 for $t < t_1$

$$\left(\frac{kx}{F_0} \right) = \frac{1}{1-r^2} [\sin p t - r \sin \omega_n t]$$

Differentiate and set equal to zero for max.

$$\left(\frac{k\dot{x}}{F_0} \right) = p \cos p t_m - r \omega_n \cos \omega_n t_m = p [\cos p t_m - \cos \omega_n t_m] = 0$$

Use identity

$$\cos p t_m - \cos \omega_n t_m = -2 \sin \frac{1}{2}(p + \omega_n) t_m \cdot \sin \frac{1}{2}(p - \omega_n) t_m = 0$$

which is satisfied when $\frac{1}{2}(p + \omega_n) t_m = n\pi$, $n=1, 2, \dots$

Therefore the time t_m corresponding to $\left(\frac{kx}{F_0} \right)_{\max}$ is

$$t_m = \frac{2n\pi}{p + \omega_n}$$

Substituting into $\frac{kx}{F_0}$, we have

$$\left(\frac{kx}{F_0} \right)_{\max} = \frac{1}{1-r^2} \left[\sin \left(\frac{2n\pi p}{p + \omega_n} \right) - r \sin \left(\frac{2n\pi \omega_n}{p + \omega_n} \right) \right]$$

4-23 part (b) for $t > t_1$

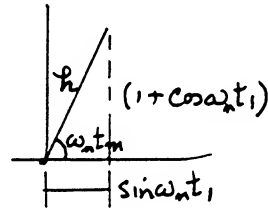
Start with the first equation 4.4-(c) 6

$$\left(\frac{kx}{F_0}\right) = \frac{-r}{1-r^2} \left\{ (1+\cos\omega_m t_1) \sin\omega_m t - \sin\omega_m t_1 \cos\omega_m t \right\}$$

where t is measured from t_1 . Differentiate and equate to zero to find t_m corresponding to $x(t)_{\max}$

$$\left(\frac{k\dot{x}}{F_0}\right) = \frac{-r\omega_m}{1-r^2} \left\{ (1+\cos\omega_m t_1) \cos\omega_m t_m - \sin\omega_m t_1 \sin\omega_m t_m \right\} = 0$$

$$\therefore \tan\omega_m t_m = \frac{(1+\cos\omega_m t_1)}{\sin\omega_m t_1}$$



$$h = \sqrt{\sin^2\omega_m t_1 + (1+\cos\omega_m t_1)^2} = \sqrt{2(1+\cos\omega_m t_1)}$$

$$= \sqrt{4\cos^2\frac{\omega_m t_1}{2}} = 2\cos\frac{\omega_m t_1}{2}$$

$$\therefore \sin\omega_m t_m = \frac{1+\cos\omega_m t_1}{2\cos\frac{\omega_m t_1}{2}}, \quad \cos\omega_m t_m = \frac{\sin\omega_m t_1}{2\cos\frac{\omega_m t_1}{2}}$$

Substitute into first eq. for $\left(\frac{kx}{F_0}\right)_{\max}$

$$\left(\frac{kx}{F_0}\right)_{\max} = \frac{-r}{1-r^2} \left\{ \frac{(1+\cos\omega_m t_1)^2 + \sin^2\omega_m t_1}{2\cos\frac{\omega_m t_1}{2}} \right\}$$

$$= \frac{-r}{1-r^2} \left\{ \frac{2(1+\cos\omega_m t_1)}{2\cos\frac{\omega_m t_1}{2}} \right\} = \frac{-r}{1-r^2} \left\{ \frac{2\cos^2\frac{\omega_m t_1}{2}}{\cos\frac{\omega_m t_1}{2}} \right\}$$

$$= \frac{-r}{1-r^2} \left\{ 2\cos\frac{\omega_m t_1}{2} \right\} = \frac{2}{r - \frac{1}{r}} \cos\frac{\omega_m t_1}{2}$$

$$= \left(\frac{2}{\frac{1}{2t_1} - \frac{2t_1}{\pi}} \right) \cos\frac{\pi t_1}{\pi}$$

4-23 Cont.

$$\left(\frac{kx}{F_0}\right)_{\max} = \frac{1}{1-r^2} \left[\sin\left(\frac{2n\pi r}{1+r}\right) - r \sin\left(\frac{2n\pi}{1+r}\right) \right]$$

The first term can be reduced as follows

$$\sin\left(\frac{2n\pi r}{1+r}\right) = \sin \frac{2n\pi(r+1-1)}{1+r} = \sin\left(2n\pi - \frac{2n\pi}{1+r}\right)$$

$$= \sin 2n\pi \cos \frac{2n\pi}{1+r} - \cos 2n\pi \sin \frac{2n\pi}{1+r}$$

$$= -\sin\left(\frac{2n\pi}{1+r}\right)$$

\therefore Replace 2nd term by $-\sin\left(\frac{2n\pi}{1+r}\right)$

$$\left(\frac{kx}{F_0}\right)_{\max} = \frac{(1+r)}{1-r^2} \left[\sin \frac{2n\pi r}{1+r} \right]$$

$$= \frac{1}{1-r} \sin \frac{2n\pi r}{1+r} \quad \text{where } r = \frac{\tau}{2t_1}$$

$$\left(\frac{kx}{F_0}\right)_{\max} = \frac{1}{1-\frac{\tau}{2t_1}} \sin \left[\frac{2n\pi \left(\frac{\tau}{2t_1}\right)}{1+\frac{\tau}{2t_1}} \right]$$

The above equation is plotted as Fig 4.5-5 against the horizontal scale t_1/τ . Computed values appear below

t_1/τ	$\left(\frac{kx}{F_0}\right)_{\max} \quad n=1$	t_1/τ	$\left(\frac{kx}{F_0}\right)_{\max} \quad n=2$
.10	.2165	2.0	1.268
.25	.866	2.5	1.083
.30	1.063	3.0	1.170
.40	1.368		
.60	1.694		
1.0	1.732		
1.5	1.500		

4-24 Given vel. = $v = v_0 (u(t) - t/t_1)$

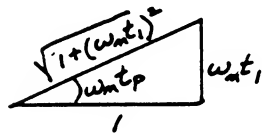
$$\text{Accel} = a = v_0 [\delta(t) - 1/t_1] = \ddot{y}(t)$$

Subst into Eq. 4.2-5

$$z = -\frac{v_0}{\omega_m} \int_0^t [\delta(\xi) - \frac{1}{t_1}] \sin \omega_m (t-\xi) d\xi \quad 0 < t < t_1,$$

$$= \frac{v_0}{\omega_m} \left\{ -\sin \omega_m t + \frac{1}{\omega_m t_1} (1 - \cos \omega_m t) \right\} \quad 0 < t < t_1,$$

$$\frac{dz}{dt} = \frac{v_0}{\omega_m} \left\{ -\omega_m \cos \omega_m t + \frac{1}{t_1} \sin \omega_m t \right\} = 0$$



$$\text{or } \tan \omega_m t_p = \omega_m t_1 \quad \text{for } z_{\max}$$

$$\sin \omega_m t_p = \frac{\omega_m t_1}{\sqrt{1 + (\omega_m t_1)^2}}$$

$$\cos \omega_m t_p = \frac{1}{\sqrt{1 + (\omega_m t_1)^2}}$$

$$\therefore \frac{z_{\max} \omega_m}{v_0} = \left\{ \frac{1}{\omega_m t_1} - \frac{1}{\omega_m t_1 \sqrt{1 + (\omega_m t_1)^2}} - \frac{\omega_m t_1}{\sqrt{1 + (\omega_m t_1)^2}} \right\}$$

4-25

If $t > t_1$, the solution is

$$z = -\frac{v_0}{\omega_m} \int_0^{t_1} [\delta(\xi) - \frac{1}{t_1}] \sin \omega_m (t-\xi) d\xi$$

$$= \frac{v_0}{\omega_m} \left\{ -\sin \omega_m t + \frac{1}{\omega_m t_1} [\cos \omega_m (t-t_1) - \cos \omega_m t] \right\}$$

4-26 $\ddot{z} + \omega_m^2 z = -\ddot{y} = 100 - 20\delta(t)$

$$\tau = \frac{2\pi}{\omega_m} = 0.628 \text{ s.}$$

choose $\Delta t = 0.05 \text{ s}$

For the impulse response, it is advisable to treat it separately.

Its response is $-\frac{20}{\omega_m} \int_0^t \delta(\xi) \sin \omega_m(t-\xi) d\xi = -2 \sin 10t$

Remaining solution can be computed numerically from D.E

$$\left. \begin{array}{l} \text{(a)} \quad \ddot{z}_i = 100 - 100 z_i \\ \text{(b)} \quad \ddot{z}_i \Delta t^2 = z_{i-1} - 2z_i + z_{i+1} \end{array} \right\} \begin{array}{l} \text{Initial cond.} \\ z_1 = \dot{z}_1 = 0 \end{array}$$

Start with $\ddot{z}_1 = 100$ from (a)

From Eq(4.7-8) $z_2 = \frac{1}{2} (0.05)^2 100 = 0.1250$

from (a) $\ddot{z}_2 = 100 - 100(0.1250)$

Find z_3 from Eq. (b) etc.

Alternately, one could use the built in ODE solver in Matlab.

```
%This is the code for problem 4-26
t0=0;
tf=.05;
x0=[0 0]';
[t,x]=ode23('prob426',t0,tf,x0);
plot(t,x)
```

```
function xdot=prob426(t,x)
xdot=[100-100*x(2);x(1)];
```

4-27

```
( ' This is example 4.7-2 on page 113 in the text' )
clear;
clg;
J=0;
T=0;
X=0;
Xdd=0;
F=0;
dt=0.05;
T(1)=0;
F(1)=0;
X(1)=0;
J(1)=1;
for I=2:25,
    J(I)=I;
    T(I)=T(1)+dt*(I-1);
    if I > 2
        if I < 5
            F(I)=500*dt*(I-1);
        else
            if I < 9
                F(I)=200-500*dt*(I-1);
            else
                F(I)=0;
            end
        end
        X(I)=Xdd(I-1)*dt^2-X(I-2)+2*X(I-1);
        Xdd(I)=2*F(I)-16*pi^2*X(I);
    else
        F(I)=500*dt*(I-1);
        X(I)=dt^2*F(I)/(3+8*pi^2*dt^2);
        Xdd(I)=2*F(I)-16*pi^2*X(I);
    end
end
A=[ '      J      ' '      TIME      ' '      DISPL      ' '      ACCLRN      ' '      FORCE      ' ]
B=[J' T' X' Xdd' F']
plot(T,X)
hold on
plot(T,X,'x')
xlabel('Time, seconds')
ylabel('Displacement X, m')
hold off
```

4-28 See Ex. 4.2-2 for $y = v_0 e^{-0.10t}$ applies for $\omega_n t_0$,
 Fig P4-28 is for $y = 60 e^{-0.10t}$ or $\omega_n t_0$

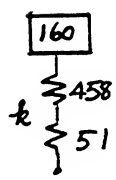
$$\therefore v_0 = 60, \quad t_0 = 10$$

For large $\omega_n t_0$ $\left(\frac{2Z}{v_0 t_0}\right)_{\max} \approx \frac{2}{\omega_n t_0}$ a rectangular hyperbola

at $\omega_n t_0 = 100$ Fig 4-28 gives $\left(\frac{2Z}{v_0 t_0}\right)_{\max} = 0.02$

For small $\omega_n t_0$ $\left(\frac{2Z}{v_0 t_0}\right)_{\max} \approx 1.0$

4-29



$$k = \frac{k_1 k_2}{k_1 + k_2} = \frac{458 \times 51}{458 + 51} = 45.89 \text{ #/in}$$

$$\delta_{st} = \frac{160}{45.89} = 3.49''$$

$$\frac{\ddot{x}}{g} = \sqrt{\frac{2h}{\delta_{st}}} + 1 = 1.65$$

4-30

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{458 \times 386}{160}} = \sqrt{1105} = 33.24 = \frac{2\pi}{T}$$

$$\therefore T = \frac{2\pi}{33.24} = .1889, \quad \frac{t_1}{T} = .85$$

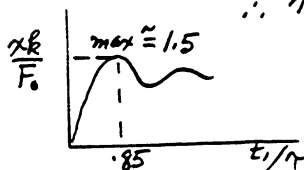
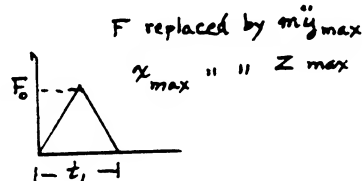
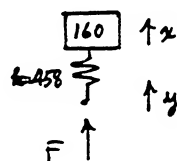


FIG. 4.5-6



4-31 $m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_n t$

With zero initial conditions, response may be evaluated from (1)

(2) or (3) below

$$(1) \quad \bar{x}(s) = \frac{\omega_n F_0 / m}{(s^2 + \omega_n^2)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$(2) \quad x(t) = X_1 e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\omega_n t + \phi_1) - \frac{F_0 \cos \omega_n t}{c\omega_n} \quad \text{Eq. (3.1-11)}$$

$$(3) \quad x(t) = F_0 \int_0^t \sin \omega_n(t-\gamma) \cdot \frac{e^{-\zeta\omega_n \gamma}}{m\omega_n \sqrt{1-\zeta^2}} \sin \sqrt{1-\zeta^2}\omega_n \gamma \, d\gamma \quad \text{Eq. (4.2-2)}$$

Result is:

$$x(t) = \frac{F_0}{c\omega_n} \left\{ \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\omega_n t + \sin^{-1}\sqrt{1-\zeta^2}) - \cos \omega_n t \right\}$$

4-32 For small damping $\delta \approx 2\pi\zeta$

At time $t = \frac{1}{f_n \delta}$, $\zeta \omega_n t = \frac{2\pi\zeta}{\delta} \approx 1.0$ Subst. into Eq. above

$$x = \frac{F_0}{c\omega_n} \left\{ e^{-1} \sin(\omega_n t + 90^\circ) - \cos \omega_n t \right\}$$

$$= \frac{F_0}{c\omega_n} \left\{ e^{-1} - 1 \right\} \cos \omega_n t = (1 - e^{-1}) \left(\frac{F_0}{c\omega_n} \cos \omega_n t \right)$$

where steady state sol. = $\left(\frac{F_0}{c\omega_n} \cos \omega_n t \right)$

66

4-33 Under harmonic force of freq. ω_n , the steady state oscillation is

$$x(t) = \frac{F_0}{c\omega_n} \cos \omega_n t \quad x(0) = \frac{F_0}{c\omega_n}, \quad \dot{x}(0) = 0$$

Transient sol. is

$$x(t) = X_1 e^{-\zeta \omega_n t} \sin(\sqrt{1-\zeta^2} \omega_n t + \phi_1) \approx X_1 e^{-\zeta \omega_n t} \sin(\omega_n t + \phi_1)$$

for ζ small

$$x(0) = X_1 \sin \phi_1 = \frac{F_0}{c\omega_n}$$

$$\dot{x}(0) = X_1 [\omega_n \cos \phi_1 - \zeta \omega_n \sin \phi_1] \approx X_1 \omega_n \cos \phi_1 = 0$$

$$\therefore \phi_1 = 90^\circ \quad \text{and} \quad X_1 = \frac{F_0}{c\omega_n}$$

Then trans. sol. with above initial cond. is

$$x(t) = \frac{F_0}{c\omega_n} e^{-\zeta \omega_n t} \cos \omega_n t$$

$$\text{At } \zeta \omega_n t = \frac{2\pi\zeta}{\delta} \approx 1.0 \quad x(t) = e^{-1} \frac{F_0}{c\omega_n} \cos \omega_n t$$

see Prob 4-32

4-34

```
%      (' This is example 4.7-1          in the text')
clear; clg;
J=0;
T=0;
X=0;
dt=0.02;
X(1)=0;
Xd(1)=0;
for I=1:24;
    J(I)=I;
    T(I)=dt*(I-1);
    if I < 6
        F(I)=100;
    elseif I < 11;
        F(I)=-1000*(T(I)-0.10)+100;
    else
        F(I)=0;
    end
    Xdd(I)=0.25*F(I)-500*X(I);
    if I > 1
        X(I+1)=2*X(I)-X(I-1)+dt^2*Xdd(I);
    else
        X(2)=X(1)+dt*Xd(1)+0.5*dt^2*Xdd(1);
    end
end
J(I+1)=I+1;
T(I+1)=dt*I;
['      J      ' '      Time      ' ' Displacement']
[J' T' X']
plot(T,X,'o')
hold on
t1=0:0.02:0.1;
x1=0.05*(1-cos(22.36*t1));
plot(t1,x1)
hold on
t2=0.1:0.02:0.2;
x2=0.05*(1-cos(22.36*t2))-(0.5*(t2-0.1)-0.02236*sin(22.36*(t2-0.10)));
plot(t2,x2)
hold on
t3=0.2:0.02:0.48;
x3=0.05*(1-cos(22.36*t3))-(0.5*(t3-0.1)-0.02236*sin(22.36*(t3-0.10)))+0.5*(t3-0.2)-0.02236*sin(22.36*(t3-0.20));
plot(t3,x3)
xlabel('t, seconds')
ylabel('Displacement, m')
hold off
```

4-35 The Differential Equation for base excitation is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

let $z = x - y$

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -\ddot{y}$$

where $-\ddot{y}$ is given by

$$y = y_0 \sin \omega t$$

$$-\ddot{y} = \omega^2 y_0 \sin \omega t$$

The sine pulse terminates at $\omega t_p = \pi$. The time increment h should be chosen smaller than $t_p/10$ or $1/10(\frac{2\pi}{\omega_n})$, whichever is the smallest.

The initial conditions are

$$x(0) = X,$$

$$y(0) = 0$$

$$\dot{x}(0) = V,$$

$$\dot{y}(0) = y_0 \omega$$

$$z(0) = X,$$

$$\dot{z}(0) = V, -y_0 \omega$$

4-35 cont. The two equations given from the Taylor series are

$$(a) \quad z_2 = z_1 + \dot{z}_1 h + \frac{h^2}{2} (F_1 - \omega_n^2 z_1 - 25\omega_n \dot{z}_1)$$

$$(b) \quad z_1 = z_2 - \dot{z}_2 h + \frac{h^2}{2} (F_2 - \omega_n^2 z_2 - 25\omega_n \dot{z}_2)$$

Since the initial conditions z_1, \dot{z}_1 , and F_1 are given Eq (a) gives z_2 . Eq (b) is solved for \dot{z}_2 which is

$$(c) \quad \dot{z}_2 = \frac{z_2 - z_1 + \frac{h^2}{2} (F_2 - \omega_n^2 z_2)}{h + h^2 5 \omega_n}$$

Thus z_2 and \dot{z}_2 for the first interval can be calculated from eqs. (a) and (c)

Calculations for $z_3, \dot{z}_3, z_4, \dot{z}_4$ etc are now made with (Eq 4.7-2) and Eq (c) generalized to index i as follows

$$(d) \quad z_{i+1} = 2z_i - z_{i-1} + h^2 (F_i - \omega_n^2 z_i - 25\omega_n \dot{z}_i)$$

$$(e) \quad \dot{z}_{i+1} = \frac{z_{i+1} - z_i + \frac{h^2}{2} (F_{i+1} - \omega_n^2 z_{i+1})}{(h + h^2 5 \omega_n)}$$

The program structure is similar to problem 4-34

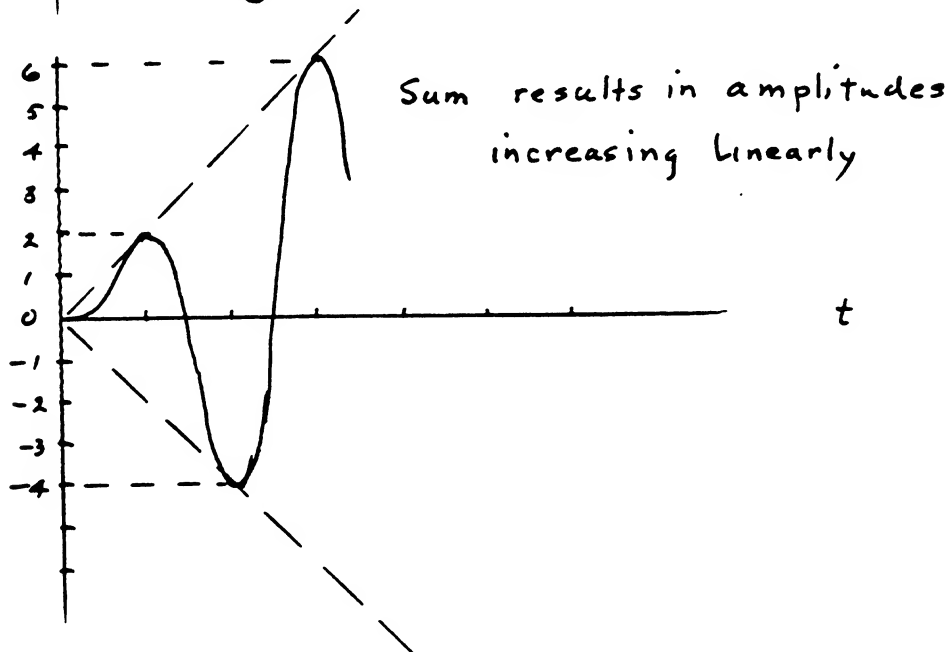
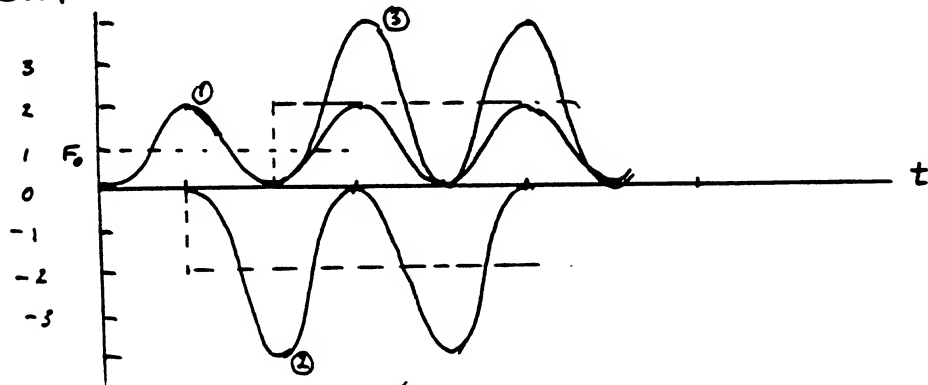
4-36 use superposition of solution to each step function

1st step of F_0 results in $x(t) = \frac{F_0}{k} (1 - \cos \omega_m t)$

2nd step of $-2F_0$ at $t = \frac{\pi}{\omega}$ adds $-\frac{2F_0}{k} [1 - \cos \omega_m (t - \frac{\pi}{\omega})]$

3rd step of $+2F_0$ at $t = \frac{2\pi}{\omega}$ adds $+\frac{2F_0}{k} [1 - \cos \omega_m (t - \frac{2\pi}{\omega})]$

etc.



4-37

$$x_{i+1} = x_i + \dot{x}_i h + \frac{\ddot{x}_i}{2} h^2 + \frac{\ddot{\ddot{x}}_i}{6} h^3 + \frac{\ddot{\ddot{\ddot{x}}}_i}{24} h^4 + \dots$$

$$x_{i-1} = x_i - \dot{x}_i h + \frac{\ddot{x}_i}{2} h^2 - \frac{\ddot{\ddot{x}}_i}{6} h^3 + \frac{\ddot{\ddot{\ddot{x}}}_i}{24} h^4 - \dots$$

$$x_{i+1} + x_{i-1} = 2x_i + \ddot{x}_i h^2 + \frac{\ddot{\ddot{\ddot{x}}}_i}{12} h^4$$

$$\ddot{x}_i = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2} - \frac{\ddot{\ddot{\ddot{x}}}_i}{12} h^2$$

$$\text{error} = -\frac{\ddot{\ddot{\ddot{x}}}_i}{12} h^2 = O(h^2)$$

4-38

t	$x = t^3$	
.8	.5120	Exact value of \dot{x} is
.9	.7290	
1.0	1.0000	$\dot{x} = 3t$
1.1	1.3310	$\therefore \dot{x}(1) = 3.0000$
1.2	1.7280	

Calculation of \dot{x} by finite difference

using $\dot{x}_i = \frac{1}{2h} (x_{i+1} - x_{i-1})$

$$h = 0.10 \quad \dot{x}_1 = \frac{1}{2 \times 0.10} (1.3310 - .7290) = 3.010$$

$$\therefore \text{Error} = .01 = .1^2 = h^2 = O(h^2)$$

$$h = 0.20 \quad \dot{x}_1 = \frac{1}{2 \times 0.20} (1.7280 - .5120) = 3.04$$

$$\therefore \text{Error} = .04 = .2^2 = h^2 = O(h^2)$$

4-39

Refer to Prob. 4-38 but use

$$\dot{x}_i = \frac{1}{h} (x_i - x_{i-1})$$

$$h = .10 \quad \dot{x}_1 = \frac{1}{.10} (1.0 - .7290) = 2.71$$

$$\therefore \text{Error} = .290 = 2.9 h = O(h)$$

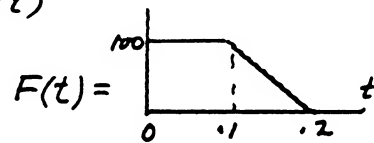
$$h = .20 \quad \dot{x}_1 = \frac{1}{.20} (1.0 - .5120) = 2.44$$

$$\therefore \text{Error} = .560 = 2.8 h = O(h)$$

4-4. C

$$\text{D.E.} \quad 4\ddot{x} + 2000x = F(t)$$

$$\therefore m = 4, \quad k = 2000$$



$$\omega_n = \sqrt{\frac{k}{m}} = 22.36$$

1st sol. due to step function (See Fig 4.7-4)

$$x_1 = \frac{F}{k} (1 - \cos \omega_n t) = \frac{100}{2000} (1 - \cos 22.36 t)$$

$$0 \leq t \leq 0.10$$

2nd sol. due to ramp function of

$$1000(t - .10) \quad \text{at } t \geq .10$$

$$\text{DE} \quad \ddot{x} + 500x = 250 t' \quad \text{where } t' = t - .10$$

$$\text{Lapl. Trans} \quad x(s) = \frac{250}{s^2(s^2 + 22.36^2)}$$

$$\begin{aligned} x_2 &= \frac{250}{22.36^2} [22.36 t' - \sin 22.36 t'] \\ &= \frac{1}{2} t' - .02236 \sin 22.36 t' \end{aligned}$$

4-40 Cont.

∴ add to first sol x_1 , the second sol. which

is $x_2 = \frac{1}{2}(t-.10) - .02236 \sin 22.36(t-.10)$

$$\text{at } t \geq .10$$

Similarly the third sol. is same as x_2
with $(t-.20)$

4-41

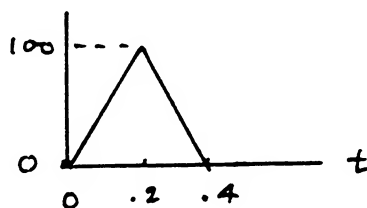
DE to be solved is

$$\ddot{x} + 16\pi^2 x = 2F(t)$$

Let $y = \dot{x}$, then

$$\dot{y} = f(x, t) = 2F(t) - 16\pi^2 x$$

where $F(t) =$



Follow Calc. procedure of Example 4.8-2

$$\omega_n = 4\pi = 12.56 = \frac{2\pi}{\gamma} \quad \gamma = .500$$

∴ suggest $h = .02$

4-41 cont

%This is the file f.m for problem 4-41

```
function [force] = f(t).
```

```
    if t<.2
```

```
        force=500*t;
```

```
    elseif t<.4
```

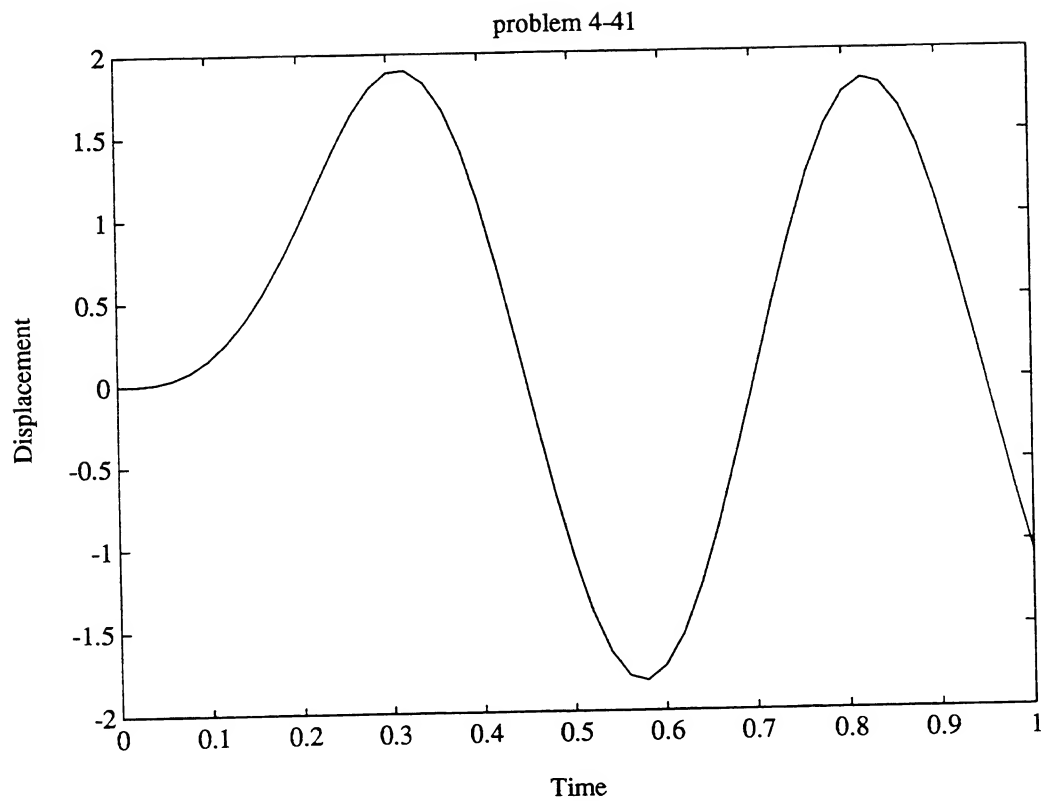
```
        force=200-500*t;
```

```
    else
```

```
        force=0;
```

```
    end
```

```
end
```



4-42

%This is the function for 4-42

```
function [force] = f(t)
```

```
    if t<.2
```

```
        force=50*t;
```

```
    elseif t<.4
```

```
        force=10;
```

```
    elseif t<.8
```

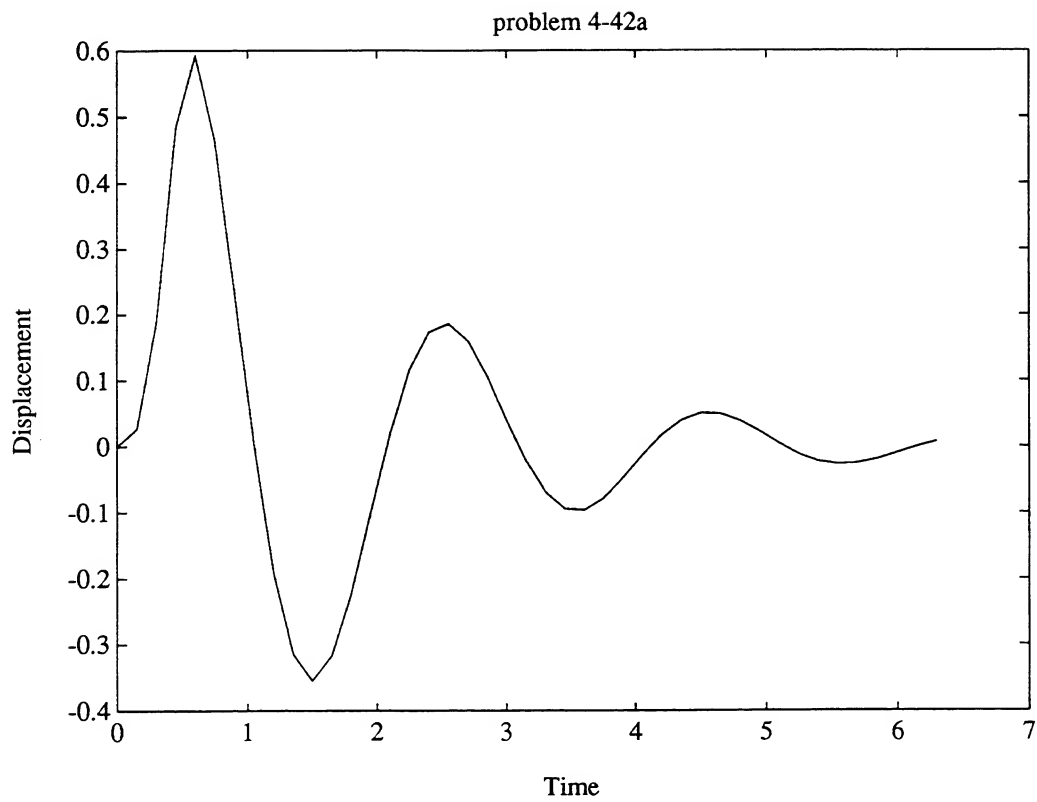
```
        force=25*(t-.8);
```

```
    else
```

```
        force=0;
```

```
    end
```

```
end
```



4-42 cont

%This is the function for 4-42b

```
function [force] = f(t)
```

```
    if t<1
```

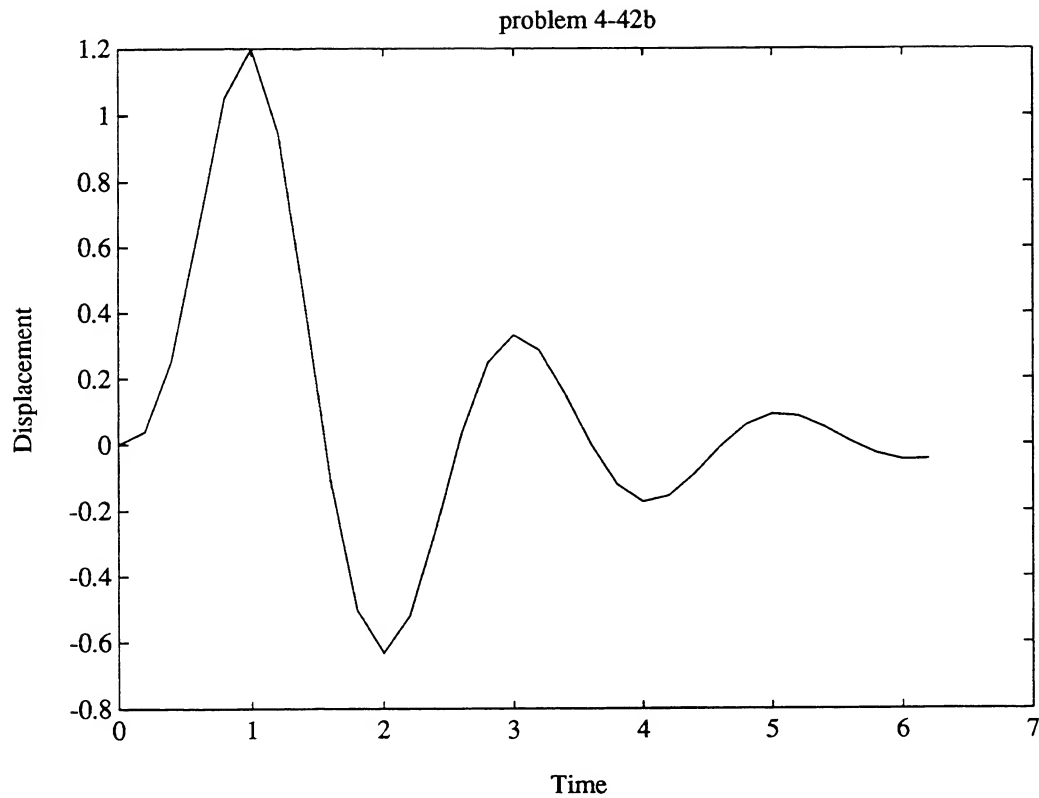
```
        force=10*sin(pi*t);
```

```
    else
```

```
        force=0;
```

```
    end
```

```
end
```



4-43

The stiffness of the crane boom is represented by k_c , measured from the extended straight line.

$$\begin{aligned}\frac{W}{g} \ddot{y} &= k_c (x - y) - W \\ \ddot{y} + \left(\frac{k_c g}{W}\right) y &= \left(\frac{k_c g}{W}\right) x - g, \quad \omega^2 = \left(\frac{k_c g}{W}\right) \\ \ddot{y} + \omega^2 y &= \omega^2 Vt - g, \quad x = Vt\end{aligned}$$

By L.T.

$$s^2 \bar{y}(s) - s y(0) - \dot{y}(0) + \omega^2 \bar{y}(s) = \frac{\omega^2 V}{s^2} - \frac{g}{s}$$

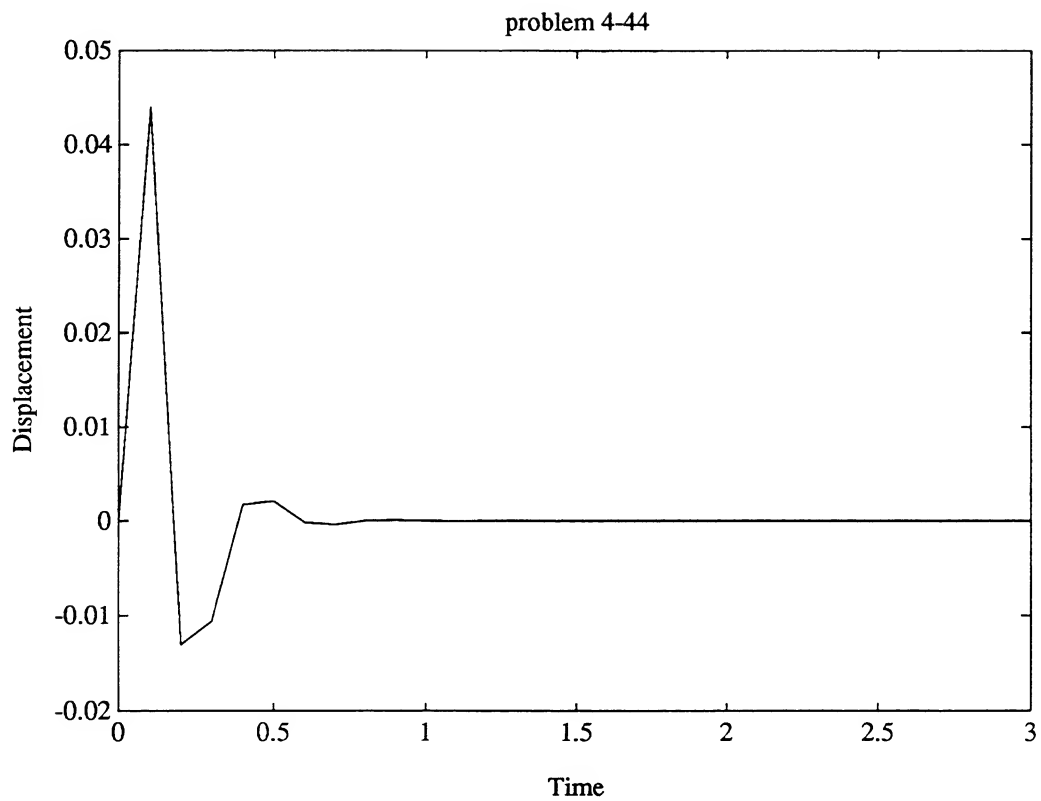
$$\bar{y}(s) = \frac{s y(0)}{s^2 + \omega^2} + \frac{\dot{y}(0)}{s^2 + \omega^2} + \frac{\omega^2 V}{s^2(s^2 + \omega^2)} - \frac{g}{s(s^2 + \omega^2)}$$

Inverse L.T.

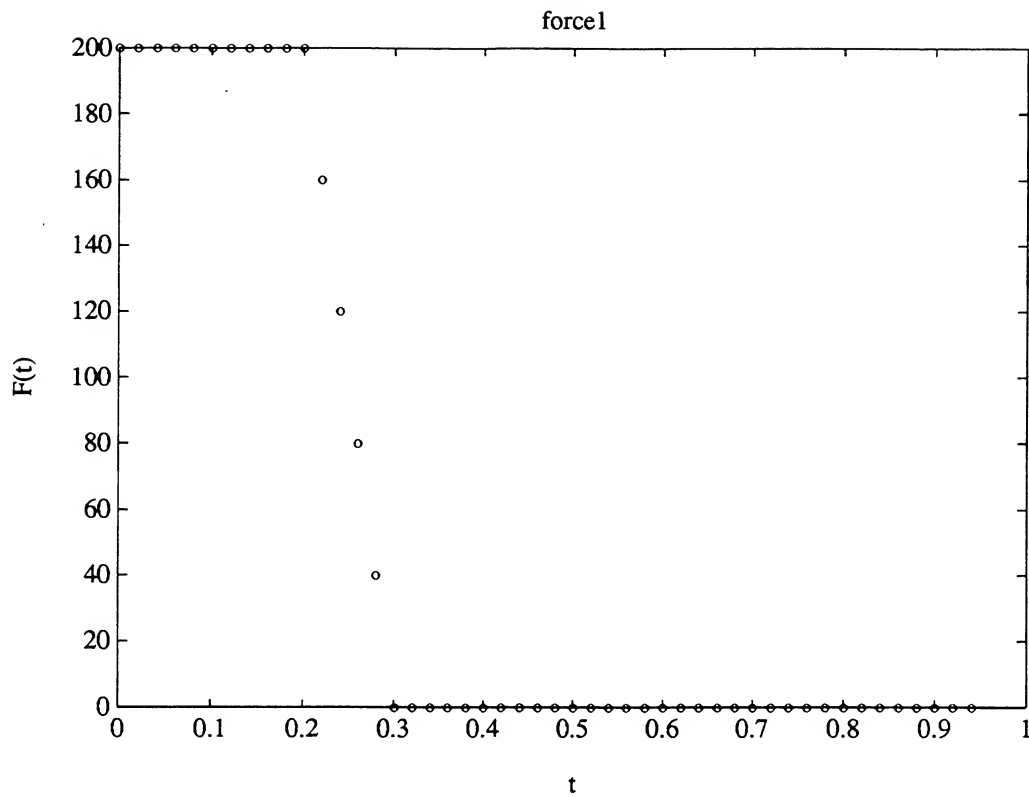
$$\begin{aligned}y(t) &= y(0) \cos \omega t + \frac{\dot{y}(0)}{\omega} \sin \omega t \\ &\quad + \frac{V}{\omega} (\omega t - \sin \omega t) - \frac{g}{\omega^2} (1 - \cos \omega t)\end{aligned}$$

4-44

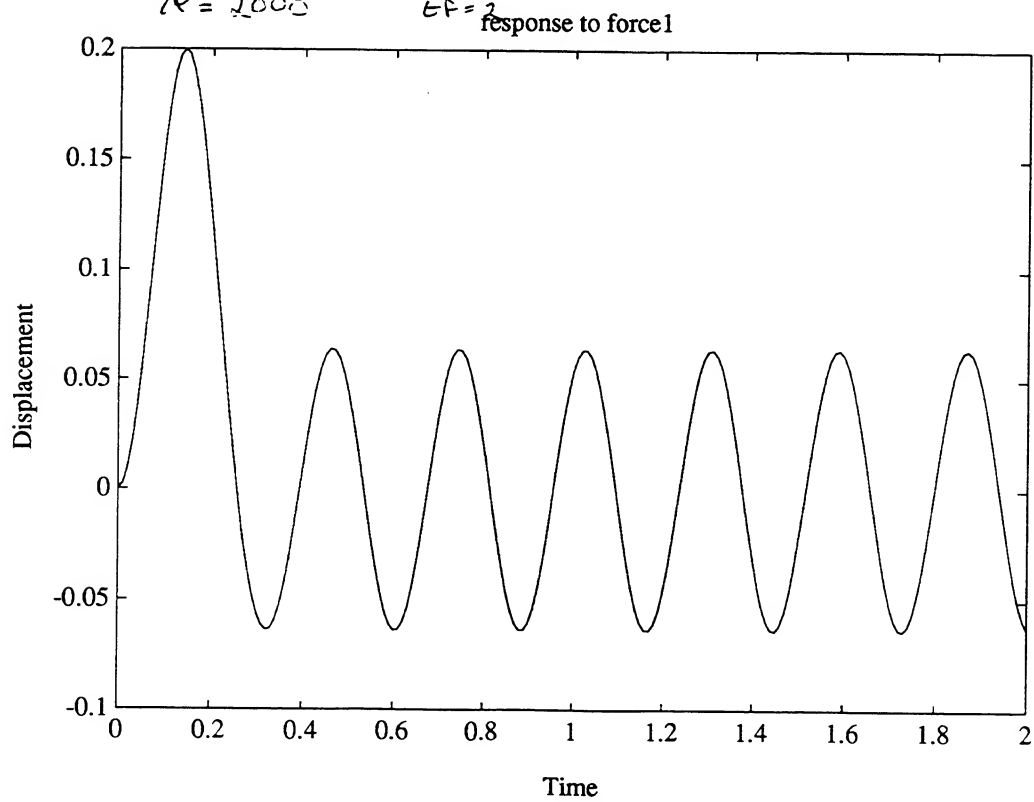
```
%This is the function for 4-44
function [force] = f(t)
    if t<.1
        force=100;
    elseif t<.2
        force=1000*t-200;
    else
        force =0;
    end
end
```



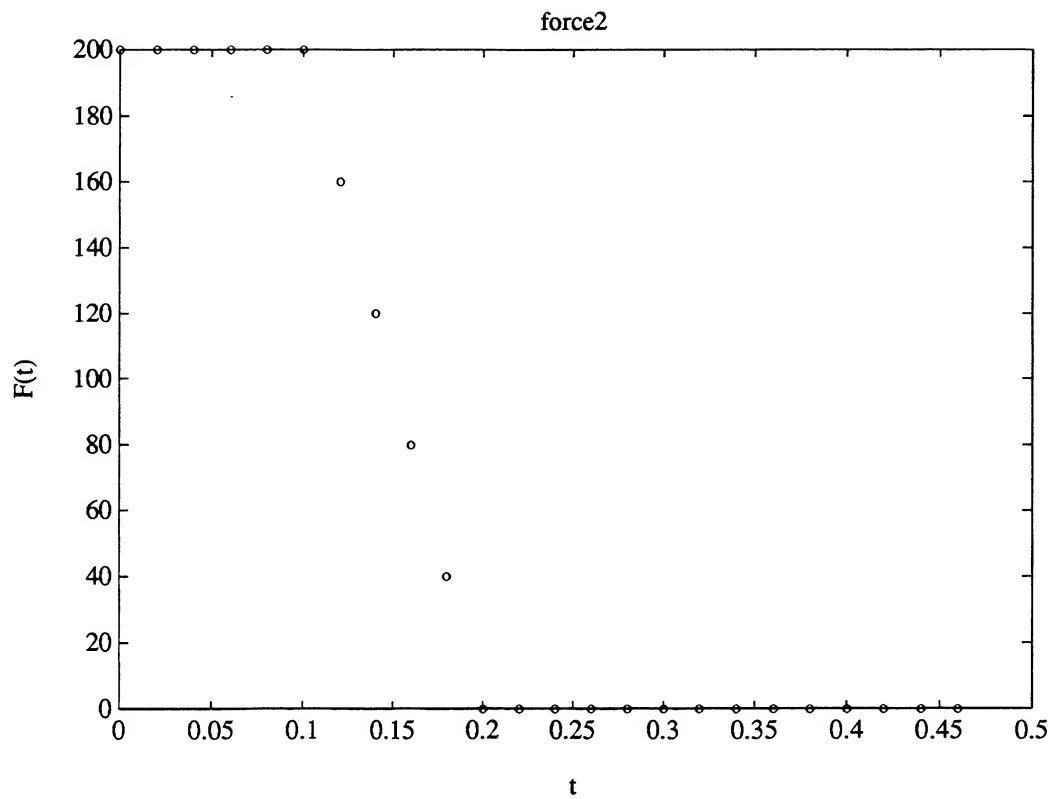
4-45



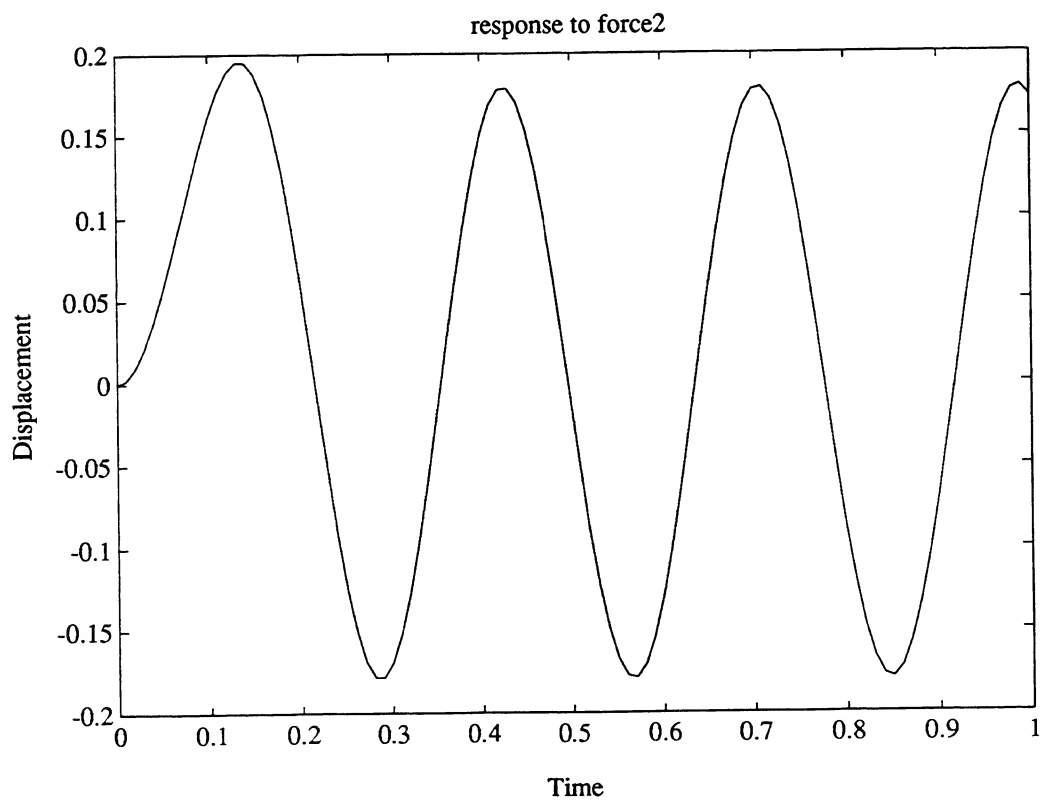
use program `runge.m`
 $m = 4$ $dt = .01$ $x_0 = \dot{x}_0 = 0$
 $c = 0$ $t_0 = 0$
 $k = 2000$ $t_f = 2$



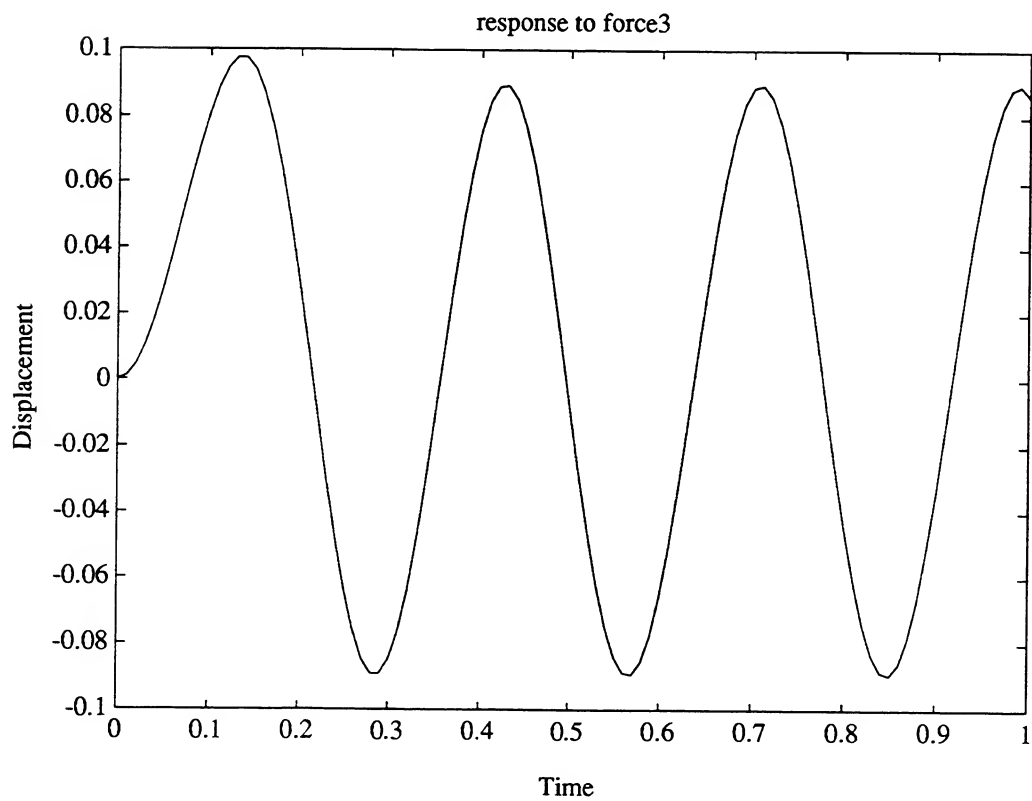
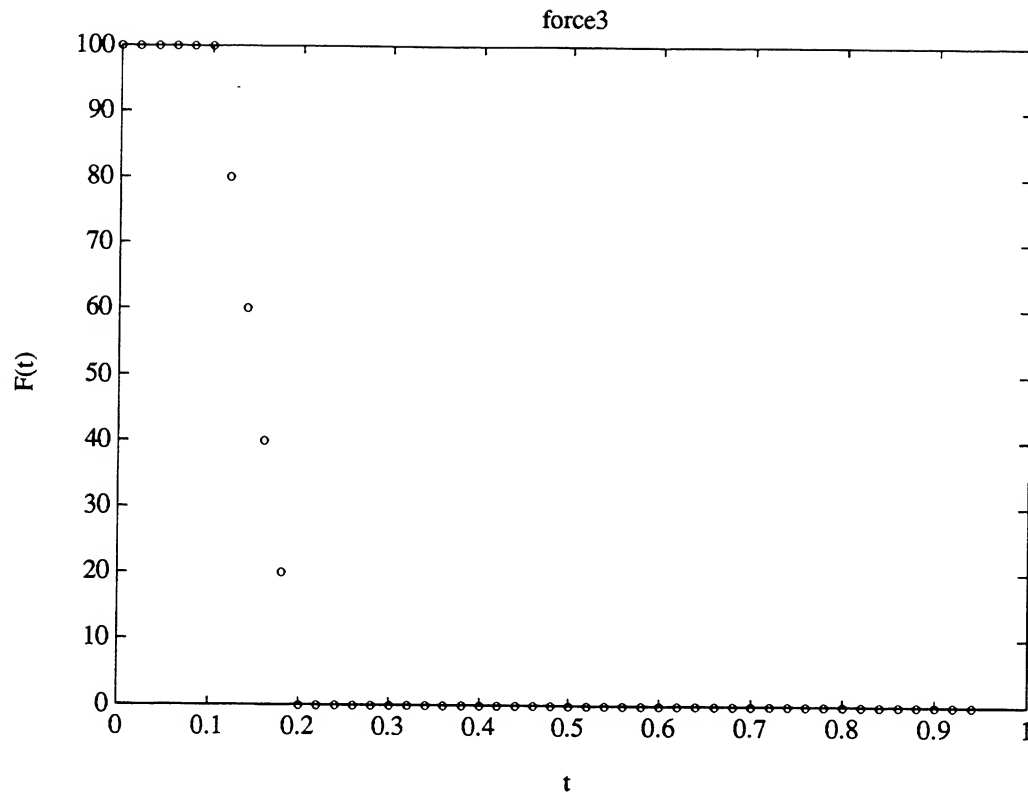
4-45 cont.



Same input as previous forcing



4-45 cont



4-46

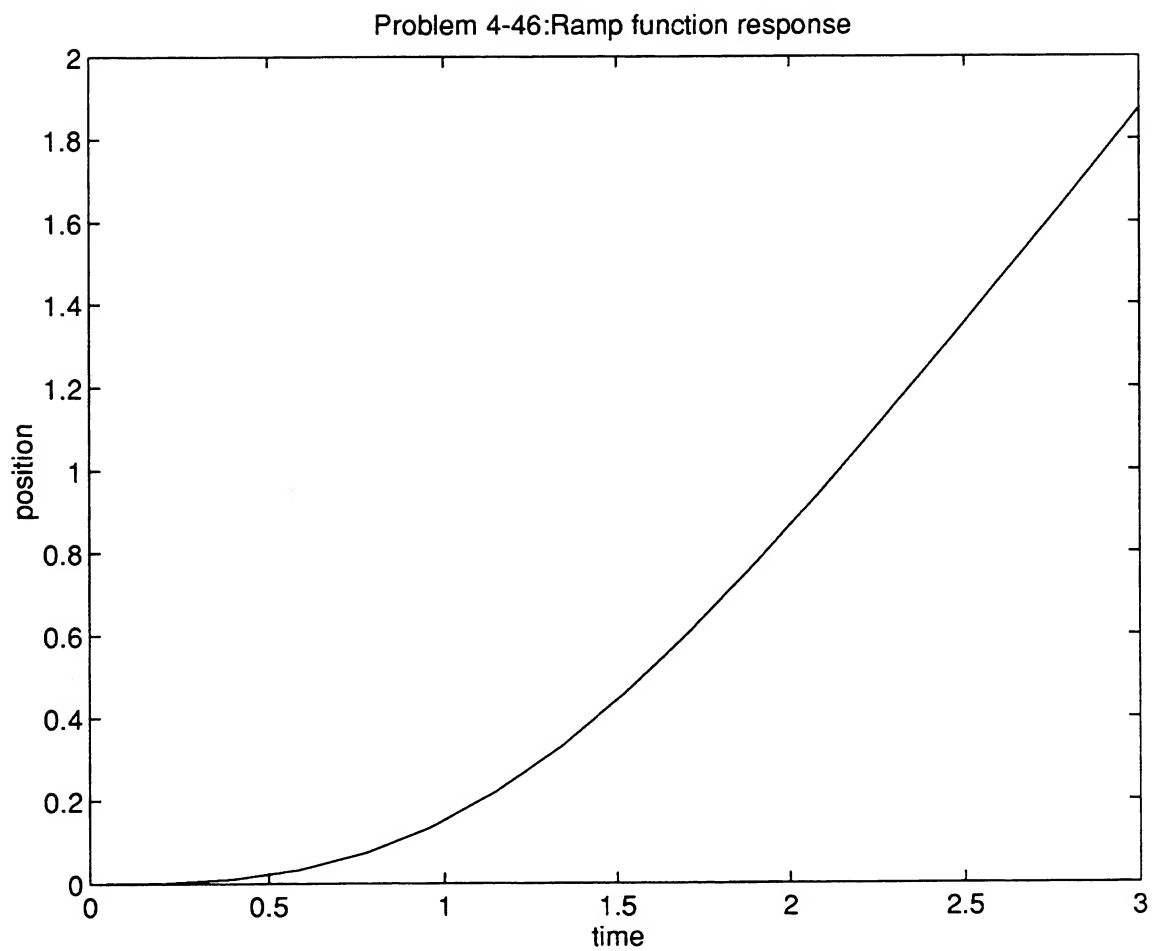
```

clear
global m k a step r
'Problem 4-46 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
k=input('Enter the value of the spring stiffness : ')
a=input('Enter the value of the damping coefficient a : ')
choice=input('Enter your choice of the forcing function function step=1 and
ramp=2 : ')
if choice==1
    step=input('Enter the value of the step function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p449s',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-4 :Step
function response')
end
if choice==2
    r=input('Enter the slope of the ramp function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p449r',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-4 :Ramp
function response')
end
function xdot=p449r(t,x)
global m k a r
if x(2)>0 | x(2)==0
    xdot=[x(2);
        (1/m)*(-a*x(2)^2-k*x(1)+r*t)];
else
    xdot=[x(2);
        (1/m)*(a*x(2)^2-k*x(1)+r*t)];
end

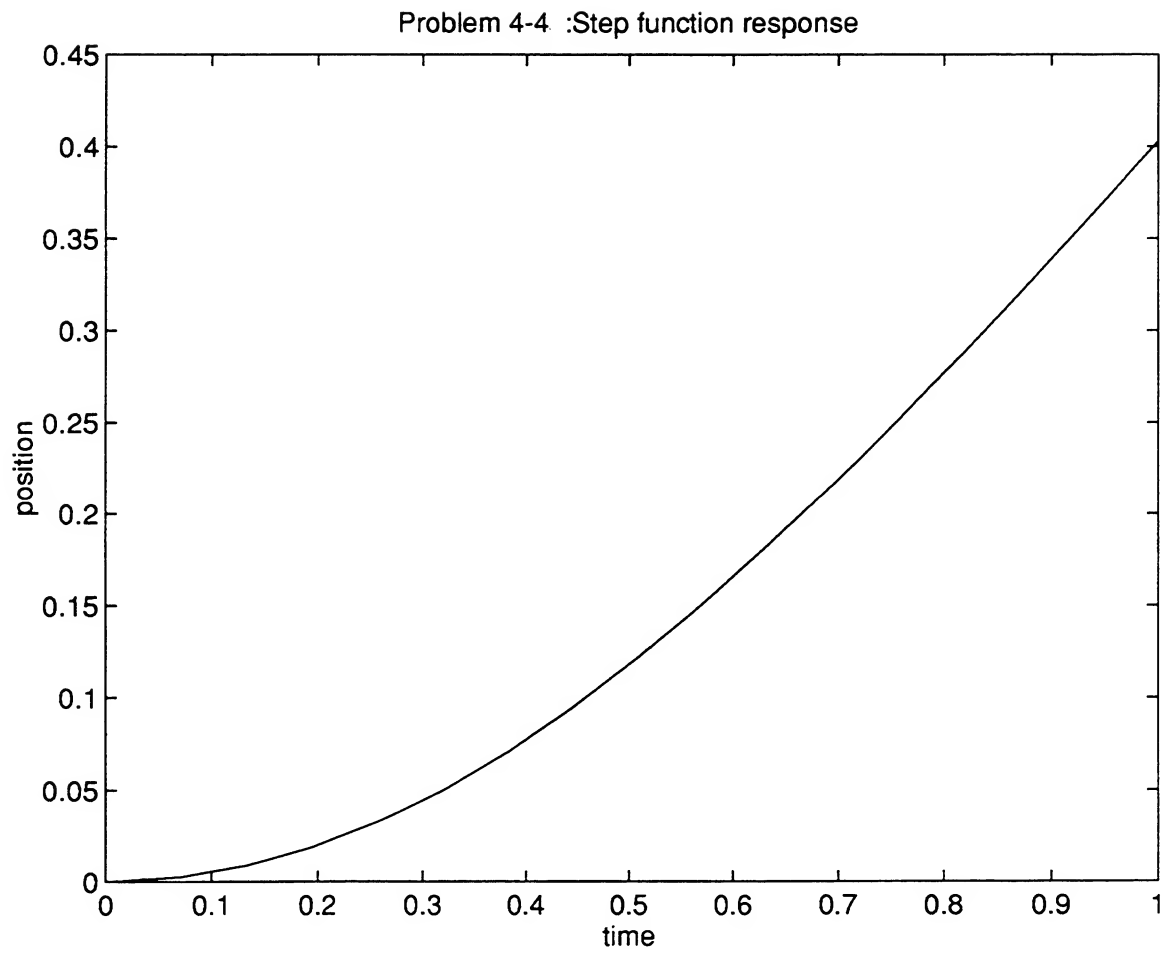
function xdot=p449s(t,x)
global m k a step
if x(2)>0 | x(2)==0
    xdot=[x(2);
        (1/m)*(-a*x(2)^2-k*x(1)+step)];
else
    xdot=[x(2);
        (1/m)*(a*x(2)^2-k*x(1)+step)];
end

```

4-46cont



4-46 cont



4-47

```
clear
global m k c mu step r
'Problem 4-47 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
c=input('Enter the value of the damping coefficient (N.s/m) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
mu=input('Enter the value of mu (N/m^3) : ')
choice=input('Enter your choice of the forcing function function step=1 and ramp=2 : ')
if choice==1
    step=input('Enter the value of the step function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447s',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Step function response')
end
if choice==2
    r=input('Enter the slope of the ramp function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447r',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Ramp function response')
end
```

```
function xdot=p447r(t,x)
global m k c mu r
xdot=[x(2);
      (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+r*t)];
```

```
function xdot=p447s(t,x)
global m k c mu step
xdot=[x(2);
      (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+step)];
```

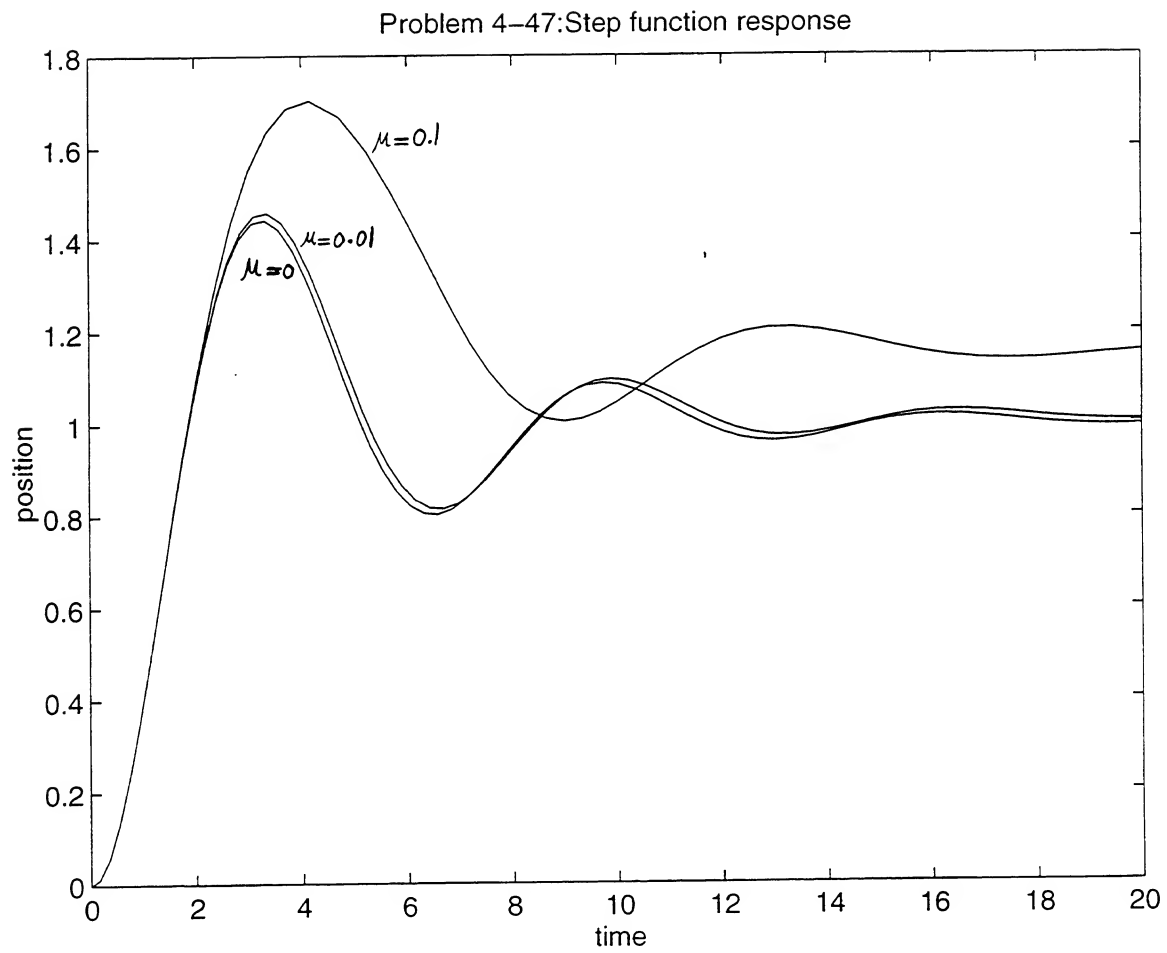
4-47

```
clear
global m k c mu step r
'Problem 4-47 Solution'
x0(1)=0;x0(2)=0;t0=0;
m=input('Enter the value of the mass (kg) : ')
c=input('Enter the value of the damping coefficient (N.s/m) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
mu=input('Enter the value of mu (N/m^3) : ')
choice=input('Enter your choice of the forcing function function step=1 and ramp=2 : ')
if choice==1
    step=input('Enter the value of the step function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447s',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Step function
response')
end
if choice==2
    r=input('Enter the slope of the ramp function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p447r',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-47:Ramp function
response')
end
```

```
function xdot=p447r(t,x)
global m k c mu r
xdot=[x(2);
      (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+r*t)];
```

```
function xdot=p447s(t,x)
global m k c mu step
xdot=[x(2);
      (1/m)*(-c*x(2)-k*x(1)+mu*x(1)^3+step)];
```

4-47 cont



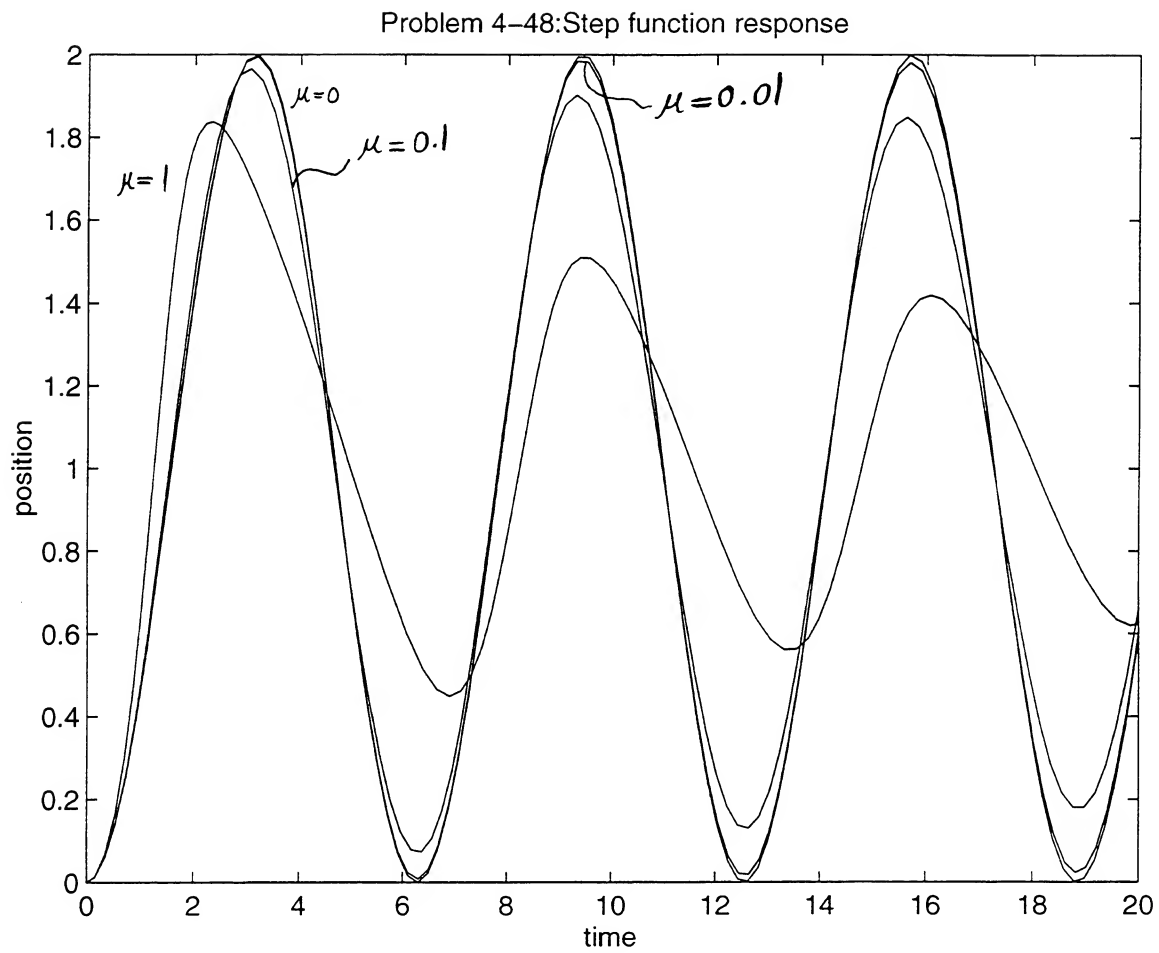
4-48

```
clear
global mu step r
'Problem 4-48 Solution'
x0(1)=0;x0(2)=0;t0=0;
mu=input('Enter the value mu : ')
choice=input('Enter your choice of the forcing function function step=1 and ramp=2 : ')
if choice==1
    step=input('Enter the value of the step function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p448s',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-48:Step function response')
end
if choice==2
    r=input('Enter the slope of the ramp function : ')
    tf=input('Enter the value of the final time : ')
    [t,x]=ode45('p448r',t0,tf,x0);
    plot(t,x(:,1));xlabel('time');ylabel('position');title('Problem 4-48:Ramp function response')
end
```

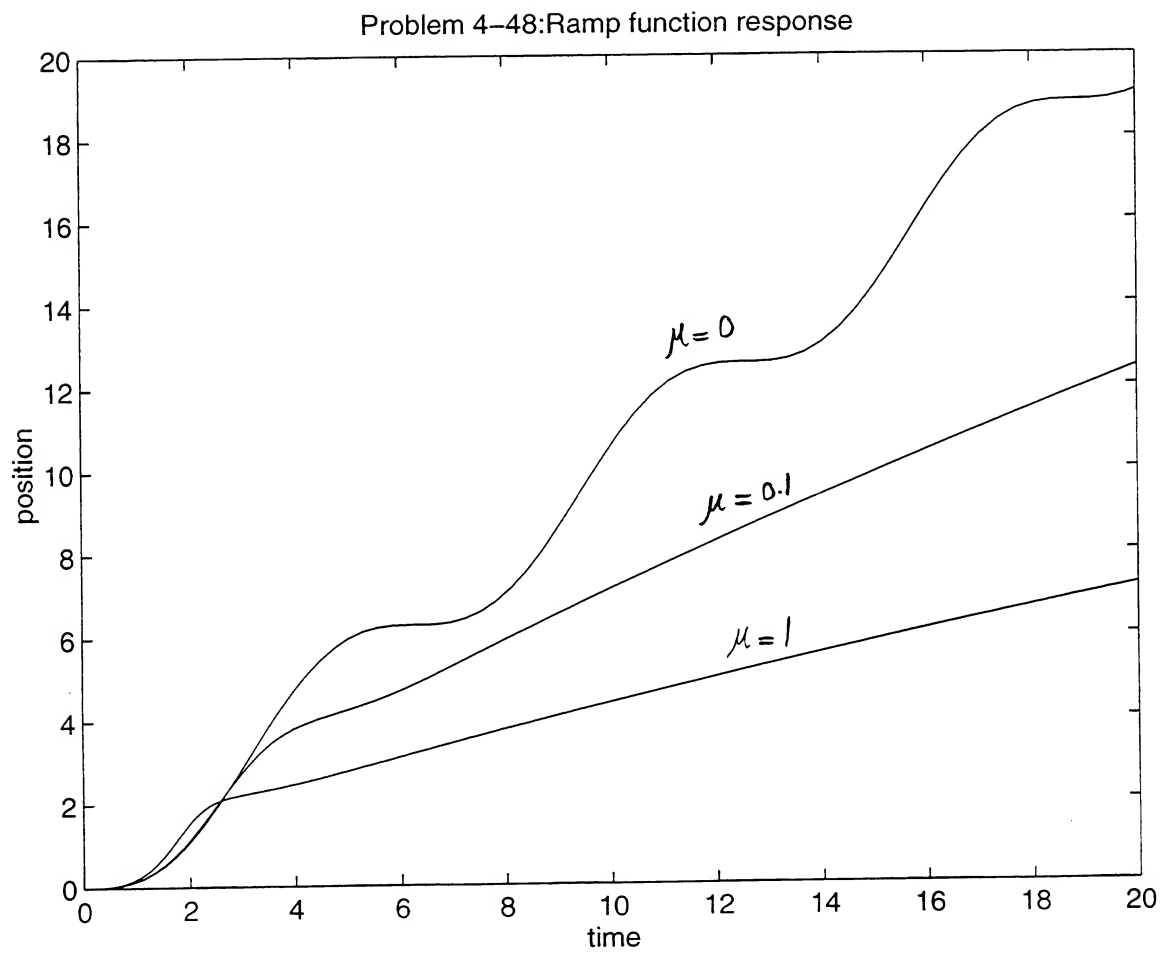
```
function xdot=p448s(t,x)
global mu step
xdot=[x(2);
      mu*x(2)*(1-x(1)^2)-x(1)+step];
```

```
function xdot=p448r(t,x)
global mu r
xdot=[x(2);
      mu*x(2)*(1-x(1)^2)-x(1)+r*t];
```

4-48 cont



4-48 cont



5-1 $m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$
 $m\ddot{x}_2 = -k(x_2 - x_1)$ origin of x_1, x_2
= equilib position

$$\begin{aligned} (2 - \frac{m\omega^2}{k})x_1 &= x_2 \\ x_1 &= (1 - \frac{m\omega^2}{k})x_2 \end{aligned} \left. \vphantom{\begin{aligned} (2 - \frac{m\omega^2}{k})x_1 &= x_2 \\ x_1 &= (1 - \frac{m\omega^2}{k})x_2 \end{aligned}} \right\} \text{let } \lambda = \frac{m\omega^2}{k}$$

characteristic eq. $\lambda^2 - 3\lambda + 1 = 0$

$$\lambda = \begin{cases} 0.382 & = \frac{m\omega_1^2}{k} & (x_1/x_2)_1 = 1 - \lambda_1 = 0.614 \\ 2.618 & = \frac{m\omega_2^2}{k} & (x_1/x_2)_2 = 1 - \lambda_2 = -1.618 \end{cases}$$

5-2 & 3 $\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k+nk) & -nk \\ -nk & (k+nk) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Let $\lambda = \frac{m\omega^2}{k}$

$$\begin{vmatrix} (1+n-\lambda) & -n \\ -n & (1+n-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 2(1+n)\lambda + (1+2n) = 0$$

$$\lambda = (1+n) \pm n, \quad x_1/x_2 = \frac{1+n-\lambda}{n}$$

For $n=1$, $\lambda_1 = 1 = \frac{m\omega_1^2}{k} \quad (x_1/x_2)_1 = 1$
 $\lambda_2 = 3 = \frac{m\omega_2^2}{k} \quad (x_1/x_2)_2 = -1$

5-4 $m \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$\lambda = \frac{m\omega^2}{k}$

$$\begin{vmatrix} (2-3\lambda) & -1 \\ -1 & (4-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - \frac{14}{3}\lambda + \frac{7}{3} = 0$$

$$\lambda = \frac{14}{6} \pm \sqrt{\left(\frac{14}{6}\right)^2 - \frac{7}{3}} = \begin{cases} 0.570 = \frac{m\omega_1^2}{k} \\ 4.096 = \frac{m\omega_2^2}{k} \end{cases}$$

$$\frac{x_1}{x_2} = (4-\lambda) \quad \left(\frac{x_1}{x_2}\right)_1 = 3.43 \quad \left(\frac{x_1}{x_2}\right)_2 = -0.096$$

5-5 $J_1 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + K \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \lambda = \frac{\omega^2 J_2}{K}$

$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0$$

$$2(1-\lambda)^2 - 1 = 0$$

$$(1-\lambda) = \pm 1/\sqrt{2}$$

$$\lambda = 1 \mp 1/\sqrt{2}$$

$$\lambda_{1,2} = \begin{cases} .293 = \frac{J_2 \omega_1^2}{K} & \left(\frac{\theta_1}{\theta_2}\right)_1 = 0.707 & \left(\frac{\theta_1}{\theta_2}\right)_2 = -0.707 \\ 1.707 = \frac{J_2 \omega_2^2}{K} \end{cases}$$

5-1

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1)$$

origin of x_1, x_2
= equilib position

$$\left. \begin{aligned} (2 - \frac{m\omega^2}{k})x_1 &= x_2 \\ x_1 &= (1 - \frac{m\omega^2}{k})x_2 \end{aligned} \right\} \text{let } \lambda = \frac{m\omega^2}{k}$$

characteristic eq. $\lambda^2 - 3\lambda + 1 = 0$

$$\lambda = \begin{cases} 0.382 & = \frac{m\omega_1^2}{k} & (x_1/x_2)_1 = 1 - \lambda_1 = 0.614 \\ 2.618 & = \frac{m\omega_2^2}{k} & (x_1/x_2)_2 = 1 - \lambda_2 = -1.618 \end{cases}$$

5-2 & 3

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k+nk) & -nk \\ -nk & (k+nk) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \frac{m\omega^2}{k}$

$$\begin{vmatrix} (1+n-\lambda) & -n \\ -n & (1+n-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 2(1+n)\lambda + (1+2n) = 0$$

$$\lambda = (1+n) \pm n, \quad x_1/x_2 = \frac{1+n-\lambda}{n}$$

For $n=1$, $\lambda_1 = 1 = \frac{m\omega_1^2}{k} \quad (x_1/x_2)_1 = 1$
 $\lambda_2 = 3 = \frac{m\omega_2^2}{k} \quad (x_1/x_2)_2 = -1$

5-4

$$m \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\lambda = \frac{m\omega^2}{k}$

$$\begin{vmatrix} (2-3\lambda) & -1 \\ -1 & (4-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - \frac{14}{3}\lambda + \frac{7}{3} = 0$$

$$\lambda = \frac{14}{6} \pm \sqrt{\left(\frac{14}{6}\right)^2 - \frac{7}{3}} = \begin{cases} 0.570 = \frac{m\omega_1^2}{k} \\ 4.096 = \frac{m\omega_2^2}{k} \end{cases}$$

$$\frac{x_1}{x_2} = (4-\lambda) \quad \left(\frac{x_1}{x_2}\right)_1 = 3.43 \quad \left(\frac{x_1}{x_2}\right)_2 = -0.096$$

5-5

$$J_2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + K \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\lambda = \frac{\omega^2 J_2}{K}$$

$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0$$

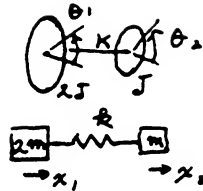
$$2(1-\lambda)^2 - 1 = 0$$

$$(1-\lambda) = \pm 1/\sqrt{2}$$

$$\lambda = 1 \mp 1/\sqrt{2}$$

$$\lambda_{1,2} = \begin{cases} .293 = \frac{J_2 \omega_1^2}{K} & \left(\frac{\theta_1}{\theta_2}\right)_1 = 0.707 & \left(\frac{\theta_1}{\theta_2}\right)_2 = -0.707 \\ 1.707 = \frac{J_2 \omega_2^2}{K} \end{cases}$$

5-6



for $K_1 = 0$

$$\begin{vmatrix} (1-2\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0$$

$$\lambda(2\lambda-3) = 0$$

$$\therefore \lambda = 0, \lambda = 1.5$$

$$\frac{\theta_1}{\theta_2} = -\frac{1}{2}$$

free rotation

In general

$$J_1 \ddot{\theta}_1 - K_2 (\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + K_2 (\theta_2 - \theta_1) = 0$$

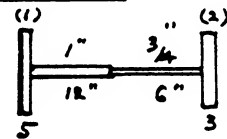
$$(\ddot{\theta}_2 - \ddot{\theta}_1) + \left(\frac{K_2}{J_2} + \frac{K_2}{J_1} \right) (\theta_2 - \theta_1) = 0$$

$$\text{Let } \phi = \theta_2 - \theta_1$$

$$\ddot{\phi} + K_2 \left(\frac{J_1 + J_2}{J_1 J_2} \right) \phi = 0$$

$$\omega_n = \sqrt{K_2 \left(\frac{J_1 + J_2}{J_1 J_2} \right)}$$

5-7



$$K_1 = \frac{G J_P}{L} = \frac{(11.5 \times 10^6)}{12} \frac{\pi (1)^4}{32} = 0.0941 \times 10^6$$

$$K_2 = \frac{(11.5 \times 10^6)}{6} \frac{\pi (\frac{3}{4})^4}{32} = 0.0595 \times 10^6$$

$$K_{eff} = \frac{K_1 K_2}{K_1 + K_2} = 0.0365 \times 10^6$$

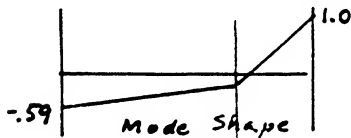
$$\omega_n = \sqrt{K_{eff} \left(\frac{J_1 + J_2}{J_1 J_2} \right)} = \sqrt{\frac{8}{15} \times 0.0365 \times 10^6} = 139.4 \text{ rad/s.}$$

characteristic eq.

$$\begin{vmatrix} (1 - \frac{\omega^2 J_1}{K_{eff}}) & -1 \\ -1 & (1 - \frac{\omega^2 J_2}{K_{eff}}) \end{vmatrix} = 0 \quad \left(\frac{\theta_1}{\theta_2} \right) = \left(1 - \frac{\omega^2 J_2}{K_{eff}} \right) = 1 - 139.4^2 \frac{3}{0.0365 \times 10^6}$$

$$= 1 - 1.5979 = -0.5979$$

Twist in each shaft \propto to $\frac{1}{K_i}$

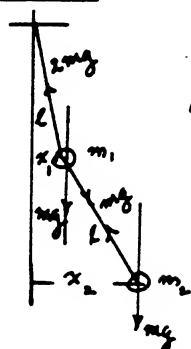


5-8

Total stiffness between cars = 16,000 #/in. System has node at middle of spring \therefore oscillates as two single deg. freed. system with each spring of $2k = 32,000$ #/in and mass m

$$\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{k \left(\frac{m_1 + m_2}{m_1 m_2} \right)} = \sqrt{\frac{2 \times 16,000 \times 386}{50,000}} = 15.72 \text{ rad/s.}$$

5-9



For small amplitudes angles are $\frac{x_1}{l}$ and $\frac{x_2 - x_1}{l}$.
Tensions are approx $2mg$ and mg .

Use $\sum F_x$

$$m\ddot{x}_1 = -2mg\left(\frac{x_1}{l}\right) + mg\left(\frac{x_2 - x_1}{l}\right)$$

$$m\ddot{x}_2 = -mg\left(\frac{x_2 - x_1}{l}\right)$$

$$\text{Let } \lambda = \frac{\omega^2 l}{g}$$

$$(3 - \lambda)X_1 - X_2 = 0$$

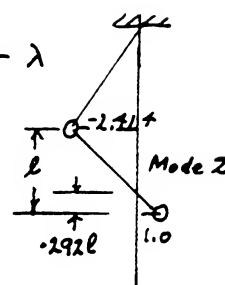
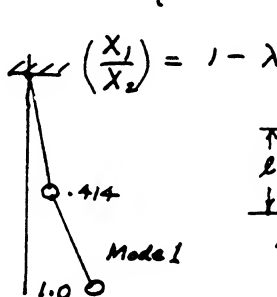
$$-X_1 + (1 - \lambda)X_2 = 0$$

$$\lambda^2 - 4\lambda + 2 = 0, \quad \lambda = 2 \pm \sqrt{2} = \begin{cases} 0.586 \\ 3.414 \end{cases}$$

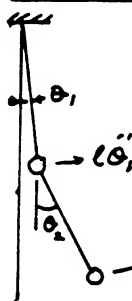
$$\omega_n = \sqrt{(2 \pm \sqrt{2}) \frac{g}{l}} = \begin{cases} 0.764 \sqrt{\frac{g}{l}} \\ 1.850 \sqrt{\frac{g}{l}} \end{cases}$$

$$\left(\frac{X_1}{X_2}\right)_1 = 1 - \lambda_1 = 0.414$$

$$\left(\frac{X_1}{X_2}\right)_2 = -2.414$$



5-10



$$ml\ddot{\theta}_1 = -2mg\theta_1 + mg\theta_2$$

$$ml(\ddot{\theta}_1 + \ddot{\theta}_2) = -mg\theta_2$$

$$\lambda = \frac{\omega^2 l}{g}$$

$$(2 - \lambda)\theta_1 - \theta_2 = 0$$

$$-\lambda\theta_1 + (1 - \lambda)\theta_2 = 0$$

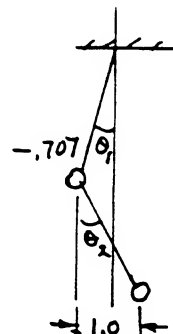
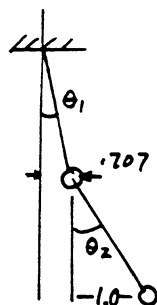
$$\lambda^2 - 4\lambda + 2 = 0$$

Same as Prob 5-9

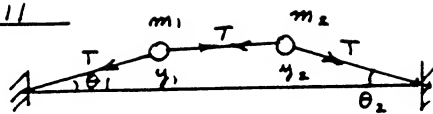
$$\frac{\theta_1}{\theta_2} = \frac{1 - \lambda}{\lambda}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.707 \\ 1.00 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} -0.707 \\ 1.00 \end{Bmatrix}$$



5-11



$$\theta_1 \approx y_1/l$$

$$\theta_2 \approx y_2/l$$

$$m_1 \ddot{y}_1 = -T \frac{y_1}{l} + T \left(\frac{y_2 - y_1}{l} \right)$$

$$m_2 \ddot{y}_2 = -T \left(\frac{y_2 - y_1}{l} \right) - T \frac{y_2}{l}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \frac{T}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

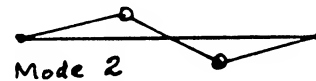
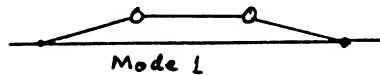
5-12

Let $\lambda = \omega^2 \frac{m l}{T}$ in Prob 5-11

$$\begin{bmatrix} (2-\lambda) & -1 \\ -1 & (2-\lambda) \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1 \quad \omega_1 = \sqrt{T/ml} \quad \left(\frac{Y_1}{Y_2} \right)_1 = 1$$

$$\lambda_2 = 3 \quad \omega_2 = \sqrt{3T/ml} \quad \left(\frac{Y_1}{Y_2} \right)_2 = -1$$



5-13

$$\begin{vmatrix} (2-2\lambda) & -1 \\ -1 & (2-\lambda) \end{vmatrix} = 0$$

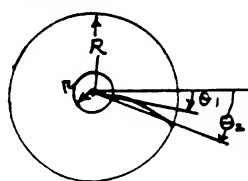
$$2\lambda^2 - 6\lambda + 3 = 0$$

$$\lambda = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 1.5}$$

$$\lambda_1 = 0.634 \quad \omega_1 = 0.796 \sqrt{T/ml} \quad \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.00 \\ 0.732 \end{Bmatrix}$$

$$\lambda_2 = 2.366 \quad \omega_2 = 1.538 \sqrt{T/ml} \quad \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -2.732 \end{Bmatrix}$$

5-14



$$J_1 \ddot{\theta}_1 = -K_1 \theta_1 + K_2 (\theta_2 - \theta_1)$$

$$J_2 \ddot{\theta}_2 = -K_2 (\theta_2 - \theta_1)$$

$$K_2 = 4 k_2 R^2$$

Defl. of spring =

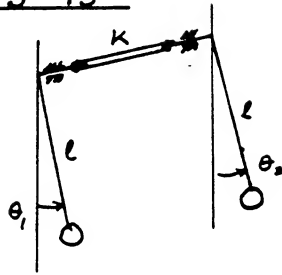
$$R(\theta_2 - \theta_1) = \frac{\text{Force}}{k_2} = \frac{F}{k_2}$$

$$\text{Torque} = R F = k_2 R^2 (\theta_2 - \theta_1)$$

$$\begin{bmatrix} \left(\frac{K_1 + K_2}{J_1} - \omega^2 \right) & -\frac{K_2}{J_1} \\ -\frac{K_2}{J_2} & \left(\frac{K_2}{J_2} - \omega^2 \right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_1} + \frac{K_2}{J_2} \right] \omega^2 + \frac{K_1}{J_1} \frac{K_2}{J_2} = 0$$

5-15



$$m l^2 \ddot{\theta}_1 = -m g l \theta_1 + K (\theta_2 - \theta_1)$$

$$m l^2 \ddot{\theta}_2 = -m g l \theta_2 - K (\theta_2 - \theta_1)$$

$$\begin{bmatrix} \left(\frac{g}{l} + \frac{K}{m l^2} - \omega^2\right) & -\frac{K}{m l^2} \\ -\frac{K}{m l^2} & \left(\frac{g}{l} + \frac{K}{m l^2} - \omega^2\right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\omega^4 - 2\left(\frac{g}{l} + \frac{K}{m l^2}\right) \omega^2 + \frac{g}{l} \left(\frac{g}{l} + 2 \frac{K}{m l^2}\right) = 0$$

$$\omega^2 = \frac{g}{l} + \frac{K}{m l^2} (1 \pm 1) \quad \frac{\theta_1}{\theta_2} = \frac{\frac{g}{l} + \frac{K}{m l^2} - \omega^2}{K/m l^2}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\text{For } \begin{cases} l = 19.3'' \\ m g = 3.86 \text{ lb} \\ K = 2.0 \text{ lb.in/rad} \end{cases} \quad \omega_1 = \sqrt{\frac{g}{l}} = 4.4721$$

$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2K}{m l^2}} = \sqrt{20 + 1.0739} = 4.5906$$

$$\text{For } \theta_1(0) = 0, \quad \theta_2(0) = \theta_0, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$$

$$\theta_1 = \frac{\theta_0}{2} \cos \omega_1 t - \frac{\theta_0}{2} \cos \omega_2 t = \theta_0 \sin \frac{1}{2}(\omega_2 - \omega_1)t \cdot \sin \frac{1}{2}(\omega_1 + \omega_2)t$$

$$\theta_2 = \frac{\theta_0}{2} \cos \omega_1 t + \frac{\theta_0}{2} \cos \omega_2 t = \theta_0 \cos \frac{1}{2}(\omega_2 - \omega_1)t \cdot \cos \frac{1}{2}(\omega_1 + \omega_2)t$$

$$\left. \begin{aligned} \theta_1 &= \theta_0 \sin 0.0593 t \cdot \sin 4.5314 t \\ \theta_2 &= \theta_0 \cos 0.0593 t \cdot \cos 4.5314 t \end{aligned} \right\} \text{ Beat period } T_b = \frac{\pi}{0.0593}$$

$$= 52.978 \text{ sec.}$$

5-16

From Prob. 5-4

$$\lambda = \begin{cases} 0.5691 & (x_1/x_2) = 3.4309 \\ 4.0972 & (x_1/x_2) = -0.0972 \end{cases}$$

$$\text{For } x_1(0) = A, \quad x_2(0) = 0$$

$$\dot{x}_1(0) = \dot{x}_2(0) = 0$$

$$x_1 = 3.4309 C_1 \cos \omega_1 t - 0.0972 C_2 \cos \omega_2 t$$

$$x_2 = 1.0000 C_1 \cos \omega_1 t + 1.0000 C_2 \cos \omega_2 t$$

i.e. Mode 1 is multiplied by C_1 and mode 2 by C_2

At $t=0$

$$A = 3.4309 C_1 - 0.0972 C_2$$

$$0 = C_1 + C_2 \quad \therefore C_1 = -C_2$$

5-16 Cont.

$$A = -3.4309 C_2 - 0.0972 C_2 = -3.5281 C_2$$

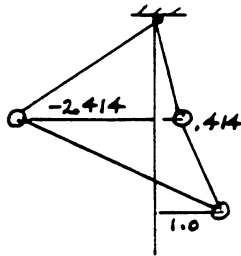
$$\therefore C_2 = -0.2834 A, \quad C_1 = 0.2834 A$$

$$x_1 = 0.9724 A \cos \omega_1 t + 0.0276 A \cos \omega_2 t$$

$$x_2 = 0.2834 A \cos \omega_1 t - 0.2834 A \cos \omega_2 t$$

5-17

Add A times mode 1 and B times mode 2



$$x_1 = 0.414 A \cos \omega_1 t - 2.414 B \cos \omega_2 t$$

$$x_2 = 1.00 A \cos \omega_1 t + 1.00 B \cos \omega_2 t$$

$$\text{At } t=0, \quad x_1(0) = x_2(0) = X$$

$$X = 0.414 A - 2.414 B$$

$$X = 1 A + 1 B \quad \therefore A = X - B$$

$$X = 0.414(X - B) - 2.414 B \quad \therefore B = -0.2072 X$$

$$A = 1.2072 X$$

$$x_1 = X \{ 0.4998 \cos \omega_1 t + 0.5002 \cos \omega_2 t \}$$

$$x_2 = X \{ 1.2072 \cos \omega_1 t - 0.2072 \cos \omega_2 t \}$$

5-18From Prob 5-1 $(x_1/x_2)_1 = 0.614, (x_1/x_2)_2 = -1.618$

$$x_1(0) = \dot{x}_1(0) = x_2(0) = 0, \quad \dot{x}_2(0) = V$$

The 4 initial conditions require 4 arbitrary constants.
We choose A times mode 1, B times mode 2 and phase angles φ_1 and φ_2

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \cos(\omega_1 t + \varphi_1) + B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \cos(\omega_2 t + \varphi_2)$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = -\omega_1 A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \sin(\omega_1 t + \varphi_1) - \omega_2 B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin(\omega_2 t + \varphi_2)$$

For $t=0$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \cos \varphi_1 + B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \cos \varphi_2$$

5-18 Cont.

$$\begin{Bmatrix} 0 \\ V \end{Bmatrix} = -\omega_1 A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \sin \varphi_1 - \omega_2 B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin \varphi_2$$

From 1st two eqs. $\varphi_1 = \varphi_2 = 90^\circ$

From 3rd eq. $\omega_1 A (0.614) = \omega_2 B (1.618)$

From 4th eq. $V = \frac{-1.618}{0.614} \omega_2 B - \omega_2 B = -3.635 \omega_2 B$

$$\therefore B = \frac{-V}{3.635 \omega_2} = -0.2751 \frac{V}{\omega_2}$$

$$A = \frac{1.618}{0.614} \frac{\omega_2}{\omega_1} \left(-0.2751 \frac{V}{\omega_2} \right) = -0.7249 \frac{V}{\omega_1}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0.7249 \frac{V}{\omega_1} \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \sin \omega_1 t + \frac{V}{3.635 \omega_2} \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin \omega_2 t$$

where: $\omega_1 = 0.618 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.618 \sqrt{\frac{k}{m}}, \quad \varphi_1 = \varphi_2 = 90^\circ$

5-19 Using same gen. eqs. of Prob. 5-18

$$\begin{Bmatrix} 0 \\ 1.0 \end{Bmatrix} = A \begin{Bmatrix} 0.614 \\ 1.000 \end{Bmatrix} \cos \varphi_1 + B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \cos \varphi_2$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = -\omega_1 A \begin{Bmatrix} 0.614 \\ 1.00 \end{Bmatrix} \sin \varphi_1 - \omega_2 B \begin{Bmatrix} -1.618 \\ 1.000 \end{Bmatrix} \sin \varphi_2$$

The second set of eqs. require $\varphi_1 = \varphi_2 = 0$

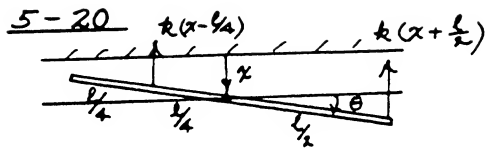
$$\therefore 0.614 A = 1.618 B \quad B = .3795 A$$

$$1.0 = A + .3795 A \quad \therefore A = .7249$$

$$B = .2751$$

Subst. into gen. eqs.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.4451 \\ 0.7249 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -0.4451 \\ 0.2751 \end{Bmatrix} \cos \omega_2 t$$



$$m\ddot{x} = -k(x + \frac{l}{2}\theta) - k(x - \frac{l}{2}\theta)$$

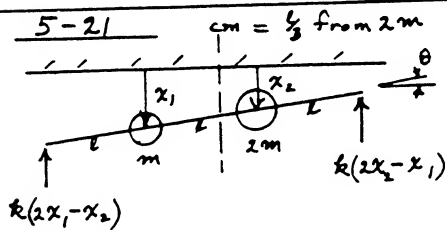
$$J\ddot{\theta} = -k(x + \frac{l}{2}\theta)\frac{l}{2} + k(x - \frac{l}{2}\theta)\frac{l}{2}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2k & k\frac{l}{4} \\ k\frac{l}{4} & k\frac{5l}{16} \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} (2 - \frac{\omega^2 m}{k}) & \frac{l}{4} \\ \frac{1}{4} & (\frac{5}{16} - \frac{\omega^2 J}{kl}) \end{vmatrix} = 0$$

$$\omega^4 \left(\frac{m}{k} \frac{J}{kl} \right) - \omega^2 \left(\frac{5}{16} \frac{m}{k} + \frac{2J}{kl} \right) + \frac{9}{16} = 0$$

$$\frac{x}{\theta} = -4 \left(\frac{5}{16} - \frac{\omega^2 J}{kl} \right)$$



$$m\ddot{x}_1 + 2m\ddot{x}_2 = -k(2x_1 - x_2) - k(2x_2 - x_1)$$

$$J_{cm} = m\left(\frac{2}{3}l\right)^2 + 2m\left(\frac{l}{3}\right)^2 = \frac{2}{3}ml^2$$

$$\theta = \frac{x_1 - x_2}{l}$$

Moment eq. $\frac{2}{3}ml^2 \left(\frac{\ddot{x}_1 - \ddot{x}_2}{l} \right) = k(2x_2 - x_1)\frac{4l}{3} - k(2x_1 - x_2)\frac{5l}{3}$

$$m \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 1 & 1 \\ 14 & -13 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{Let } \lambda = \frac{\omega^2 m}{k}$$

$$\begin{vmatrix} (1-\lambda) & (1-2\lambda) \\ (14-2\lambda) & -(13-2\lambda) \end{vmatrix} = 0$$

$$2\lambda^2 - 15\lambda + 9 = 0$$

$$\lambda_1 = 0.658 \quad \omega_1 = 0.811 \sqrt{\frac{k}{m}}$$

$$\lambda_2 = 6.842 \quad \omega_2 = 2.62 \sqrt{\frac{k}{m}}$$

$$\frac{x_1}{x_2} = \frac{(1-2\lambda)}{-(1-\lambda)} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.921 \\ 1.00 \end{Bmatrix} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.17 \\ 1.00 \end{Bmatrix}$$

5-22 $2m(\ddot{x}_1 - l\ddot{\theta}) + m\ddot{x}_2 = -k(x_1 + l\theta) - k(x_1 - 2l\theta)$

$$J_{cm}\ddot{\theta} = k(x_1 - 2l\theta)\frac{4l}{3} - k(x_1 + l\theta)\frac{5l}{3}$$

Both static & dynamic coupling present.

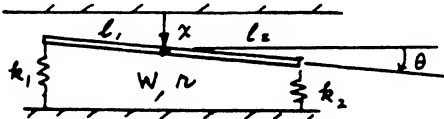
5-23 From Prob 5-9

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \frac{g}{l} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \therefore \text{Static coupling}$$

From Prob 5-10

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \frac{g}{l} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \quad \therefore \text{Static \& dynamic coupling}$$

5-24



$$\text{Let } a = \frac{1}{m} (k_1 + k_2)$$

$$b = \frac{1}{m} (k_2 l_2 - k_1 l_1)$$

$$c = \frac{1}{m n^2} (k_1 l_1^2 + k_2 l_2^2)$$

$$n = \text{rad. of gyr. about cm.}$$

$$\ddot{x} + a x + b \theta = 0$$

$$\ddot{\theta} + c \theta + \frac{b}{n^2} x = 0$$

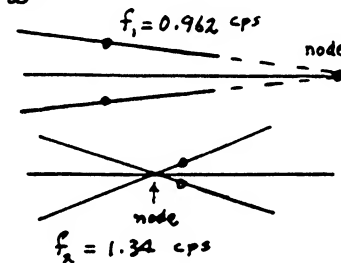
$$\omega^4 - (a+c)\omega^2 + (ac - \frac{b^2}{n^2}) = 0$$

For data given

$$a = 40.5, \quad b = 42.8, \quad c = 65.6$$

$$\omega_1^2 = 36.57 \quad \left(\frac{X}{\theta}\right)_1 = -10.9 \text{ ft/rad}$$

$$\omega_2^2 = 69.53 \quad \left(\frac{X}{\theta}\right)_2 = 1.47 \text{ ft/rad.}$$



5-25 Referring to Prob 5-24, $a = \omega_{11}^2$ is the uncoupled translational frequency squared and $c = \omega_{22}^2$ is the uncoupled rotational frequency squared. Thus the frequency equation becomes

$$\omega^4 - (\omega_{11}^2 + \omega_{22}^2)\omega^2 + (\omega_{11}^2 \omega_{22}^2 - \frac{b^2}{n^2}) = 0$$

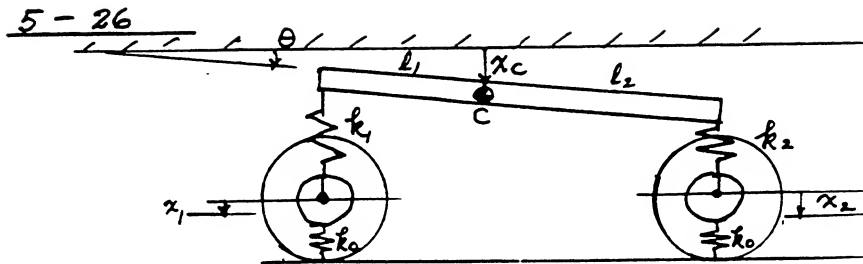
The solution is

$$\omega^2 = \frac{1}{2} (\omega_{11}^2 + \omega_{22}^2) \pm \sqrt{\frac{1}{4} (\omega_{11}^2 - \omega_{22}^2)^2 + \frac{b^2}{n^2}}$$

But b is the coupling term. If it is zero we obtain the two uncoupled frequencies. If $b \neq 0$ the radical is larger so that the uncoupled frequencies lie between the coupled frequencies

$$\text{ie. with } b=0 \quad \omega^2 = \frac{1}{2} (\omega_{11}^2 + \omega_{22}^2) \pm \frac{1}{2} (\omega_{11}^2 - \omega_{22}^2) = \begin{cases} \omega_{11}^2 \\ \omega_{22}^2 \end{cases}$$

$$\text{with } b \neq 0 \quad \omega^2 = \begin{cases} \omega_{11}^2 + \epsilon \\ \omega_{22}^2 - \epsilon \end{cases}$$



Eqs. of motion

$$m\ddot{x}_c + k_1(x_c - l_1\theta - x_1) + k_2(x_c + l_2\theta - x_2) = 0$$

$$J_c\ddot{\theta} - k_1(x_c - l_1\theta - x_1)l_1 + k_2(x_c + l_2\theta - x_2)l_2 = 0$$

$$m_0\ddot{x}_1 - k_1[x_1 - (x_c - l_1\theta)] + k_0x_1 = 0$$

$$m_0\ddot{x}_2 - k_2[x_c + l_2\theta - x_2] - k_0x_2 = 0$$

These equations can be rearranged to the matrix form given.

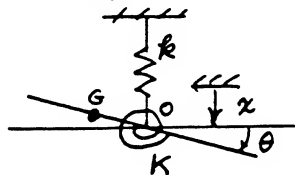
5-27

The natural frequency of the wheel-tire system is

$$\omega_n = \sqrt{\frac{k_0}{m_0}} = \sqrt{\frac{22,000 \times 32.2}{80}} = 94.10 \text{ rad/s. which is}$$

several times that of the body-spring system.

5-28



$$m(\ddot{x} - e\ddot{\theta}) + kx = 0$$

$$\sum M_G = J_G\ddot{\theta} + K\theta + kex = 0$$

$$J_0 = J_G + me^2$$

$$(J_0 - me^2)\ddot{\theta} + K\theta + kex = 0$$

5-29

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \sin \omega t$$

$$\begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 30 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \sin \omega t$$

5-29 Cont.

$$\begin{bmatrix} (30 - 0.01\omega^2) & -10 \\ -10 & (10 - 0.005\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

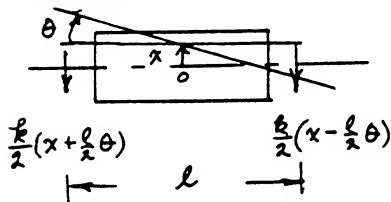
$$\omega^4 - 5000\omega^2 + 4 \times 10^6 = 0$$

$$\omega^2 = 2500 \pm \sqrt{2500^2 - 4 \times 10^6} = 2500 \pm \sqrt{2.25 \times 10^6}$$

$$\omega_1 = 31.6 \text{ rad/s} \quad \left(\frac{X_1}{X_2}\right)_1 = \frac{1}{2}$$

$$\omega_2 = 63.3 \text{ rad/s} \quad \left(\frac{X_1}{X_2}\right)_2 = -1.0$$

5-30

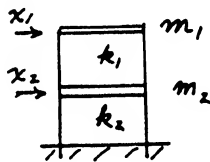


$$\Sigma F = -kx + m\omega^2 \cos \omega t = (M+m)\ddot{x} \approx M\ddot{x}$$

$$\Sigma M_0 = -\frac{k}{2}l^2\theta + m\omega^2 b \cos \omega t = J_0 \ddot{\theta}$$

$$m \ll M$$

5-31



$$m_1 \ddot{x}_1 + k_1(x_1 - x_2) = 0$$

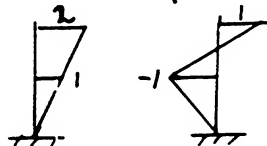
$$m_2 \ddot{x}_2 - k_1(x_1 - x_2) + k_2 x_2 = 0$$

$$\text{Let } \lambda = \frac{\omega^2 m_1}{k_1}, \quad m_2 = 2m_1, \quad k_2 = 2k_1$$

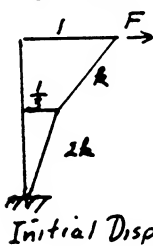
$$\begin{cases} (1-\lambda)x_1 - x_2 = 0 \\ -x_1 + (3-2\lambda)x_2 = 0 \end{cases} \quad \lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

$$\lambda = \begin{cases} \frac{1}{2} \\ 2 \end{cases} \quad \omega_1 = \sqrt{\frac{k_1}{2m_1}}, \quad \omega_2 = \sqrt{\frac{2k_1}{m_1}}$$

$$\frac{X_1}{X_2} = \frac{1}{1-\lambda} = \begin{cases} 2 \\ -1 \end{cases}$$



5-32



$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A \cos \omega_1 t + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B \cos \omega_2 t$$

$$\text{Satisfies } \dot{x}(0) = \dot{x}_2(0) = 0$$

$$\begin{Bmatrix} 1 \\ 1/3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B \quad \therefore A = \frac{4}{9}, \quad B = -\frac{1}{9}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 8/9 \\ 4/9 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} 1/9 \\ -1/9 \end{Bmatrix} \cos \omega_2 t$$

Initial Displ.

5-33 From Prob 5-32

$$x_1 = \frac{8}{9} \cos \omega_1 t + \frac{1}{9} \cos \omega_2 t$$

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$x_2 = \frac{4}{9} \cos \omega_1 t - \frac{1}{9} \cos \omega_2 t$$

$$\omega_2 = \sqrt{\frac{2k_1}{m_1}} = 2\omega_1$$

$$\left. \begin{array}{l} \text{shear in 1st story} = k_2 x_2 = 2k_1 x_2 \\ \text{" " 2nd " " } = k_1 (x_1 - x_2) \end{array} \right\} \text{Ratio} = \frac{2x_{2\max}}{(x_1 - x_2)_{\max}}$$

$$\frac{\partial x_2}{\partial t} = -\frac{4}{9} \omega_1 \sin \omega_1 t + \frac{1}{9} \omega_2 \sin \omega_2 t = 0$$

$$= -\frac{\omega_1}{9} \{ 4 \sin \omega_1 t - 2 \sin 2\omega_1 t \} = 0$$

$$= \{ 2 \sin \omega_1 t - 2 \sin \omega_1 t \cos \omega_1 t \} = 0$$

$$\therefore \cos \omega_1 t = 1, \text{ and } \omega_1 t = 0, 360^\circ \therefore x_{2\max} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

$$\frac{\partial (x_1 - x_2)}{\partial t} = -\frac{4}{9} \omega_1 \sin \omega_1 t - \frac{4}{9} \omega_1 \sin 2\omega_1 t = 0$$

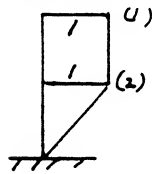
$$\text{or } \sin \omega_1 t (1 + 2 \cos \omega_1 t) = 0, \quad \cos \omega_1 t = -\frac{1}{2}, \quad \omega_1 t = 120^\circ$$

$$\therefore (x_1 - x_2)_{\max} = \frac{4}{9} \left(-\frac{1}{2}\right) + \frac{1}{9} \left(-\frac{1}{2}\right) = \frac{1}{3}$$

$$\text{Ratio of shears} = \frac{2(\frac{1}{3})}{(\frac{1}{3})} = 2$$

5-34

See Prob. 5-32



$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B$$

$$A = \frac{2}{3}, \quad B = \frac{1}{3}$$

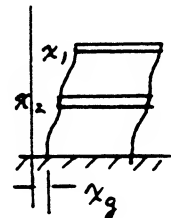
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4/3 \\ 2/3 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -1/3 \\ 1/3 \end{Bmatrix} \cos \omega_2 t$$

5-35

See Prob. 5-31

with $\lambda = \frac{\omega^2 m_1}{k_1}$

$$\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2x_g \end{Bmatrix}$$



5-33 From Prob 5-32

$$x_1 = \frac{8}{9} \cos \omega_1 t + \frac{1}{9} \cos \omega_2 t \quad \omega_1 = \sqrt{\frac{k_1}{2m_1}}$$

$$x_2 = \frac{4}{9} \cos \omega_1 t - \frac{1}{9} \cos \omega_2 t \quad \omega_2 = \sqrt{\frac{2k_1}{m_1}} = 2\omega_1$$

$$\left. \begin{array}{l} \text{shear in 1st story} = k_2 x_2 = 2k_1 x_2 \\ \text{" " 2nd " " } = k_1 (x_1 - x_2) \end{array} \right\} \text{Ratio} = \frac{2x_{2 \max}}{(x_1 - x_2)_{\max}}$$

$$\begin{aligned} \frac{\partial x_2}{\partial t} &= -\frac{4}{9} \omega_1 \sin \omega_1 t + \frac{1}{9} \omega_2 \sin \omega_2 t = 0 \\ &= -\frac{\omega_1}{9} \{ 4 \sin \omega_1 t - 2 \sin 2\omega_1 t \} = 0 \\ &= \{ 2 \sin \omega_1 t - 2 \sin \omega_1 t \cos \omega_1 t \} = 0 \end{aligned}$$

$$\therefore \cos \omega_1 t = 1, \quad \text{and } \omega_1 t = 0, 360^\circ \quad \therefore x_{2 \max} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

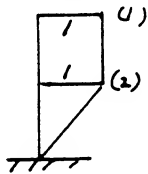
$$\begin{aligned} \frac{\partial (x_1 - x_2)}{\partial t} &= -\frac{4}{9} \omega_1 \sin \omega_1 t - \frac{4}{9} \omega_1 \sin 2\omega_1 t = 0 \\ \text{or } \sin \omega_1 t (1 + 2 \cos \omega_1 t) &= 0, \quad \cos \omega_1 t = -\frac{1}{2}, \quad \omega_1 t = 120^\circ \end{aligned}$$

$$\therefore (x_1 - x_2)_{\max} = \frac{4}{9} \left(-\frac{1}{2}\right) + \frac{2}{9} \left(-\frac{1}{2}\right) = \frac{1}{3}$$

$$\text{Ratio of shears} = \frac{2(\frac{1}{3})}{(\frac{1}{3})} = 2$$

5-34

See Prob. 5-32



$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} B$$

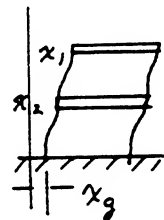
$$A = \frac{2}{3}, \quad B = \frac{1}{3}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4/3 \\ 2/3 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -1/3 \\ 1/3 \end{Bmatrix} \cos \omega_2 t$$

5-35

see Prob. 5-31 with $\lambda = \frac{\omega^2 m_1}{k_1}$

$$\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2x_g \end{Bmatrix}$$



5-35 Cont.

$$\lambda_1 = \frac{1}{2} \quad \lambda_2 = 2$$

$$\therefore \text{Denominator} = (\lambda - \frac{1}{2})(\lambda - 2) \quad \text{use Cramer's rule}$$

$$X_1 = \frac{\begin{vmatrix} 0 & -1 \\ 2 & (3-2\lambda) \end{vmatrix} X_g}{(\lambda - \frac{1}{2})(\lambda - 2)}$$

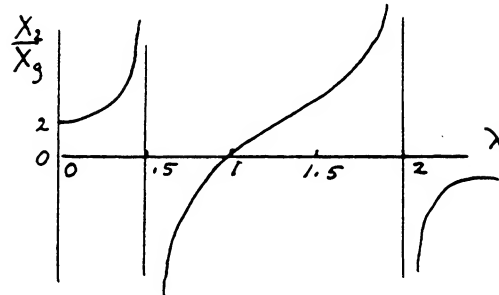
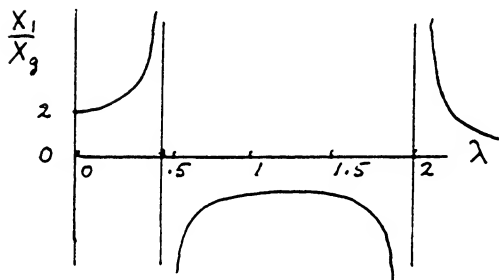
$$X_2 = \frac{\begin{vmatrix} (1-\lambda) & 0 \\ -1 & 2 \end{vmatrix} X_g}{(\lambda - \frac{1}{2})(\lambda - 2)}$$

$$\text{Response} = x_1 = X_1 \sin \omega t$$

$$x_2 = X_2 \sin \omega t$$

$$\frac{x_1}{X_g} = \frac{2 \sin \omega t}{(\lambda - \frac{1}{2})(\lambda - 2)}$$

$$\frac{x_2}{X_g} = \frac{2(1-\lambda) \sin \omega t}{(\lambda - \frac{1}{2})(\lambda - 2)}$$



5-36

$$M(\ddot{y}_0 - l_0 \ddot{\theta}) = K_k (y_G - y_0)$$

$$M \rho_c^2 \ddot{\theta} = K_k (y_G - y_0) l_0 - K_R \theta + M g l_0 \theta$$

ρ_c = rad. of gyration of body about its c.m.

5-37

$$\text{Let } \omega_h^2 = \frac{K_h}{M} \quad \omega_n^2 = \frac{K_n}{M \rho_c^2}$$

$$\lambda = \frac{\omega}{\omega_h}, \quad \left(\frac{\rho_c}{l_0}\right)^2 = \frac{1}{3}, \quad \left(\frac{\omega_n}{\omega_h}\right)^2 = 4$$

$$\begin{bmatrix} (1-\lambda^2) & \lambda^2 \\ 1 & \frac{1}{3}(4-\lambda^2) \end{bmatrix} \begin{Bmatrix} Y_0 \\ l_0 \theta_0 \end{Bmatrix} = Y_G e^{i\phi} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Characteristic eq.

$$\lambda^4 - 8\lambda^2 + 4 = 0$$

$$\lambda^2 = 4 \pm \sqrt{12} = \begin{cases} 0.54 \\ 7.46 \end{cases}$$

$$\lambda = \frac{\omega}{\omega_h} = \begin{cases} 0.734 \\ 2.73 \end{cases}$$

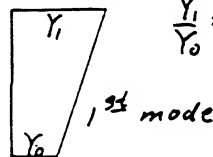
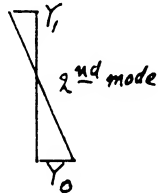
Ampl. ratio

$$\frac{1^{st} \text{ mode}}{l_0 \theta_0} \frac{Y_0}{l_0 \theta_0} = \frac{\lambda^2 - 4}{3} = -1.15$$

$$\text{Let } Y_1 = Y_0 - 2l_0 \theta_0 = \text{displ of top, then } \frac{Y_1}{l_0 \theta_0} = \frac{Y_0}{l_0 \theta_0} - 2 = -1.15 - 2 = -3.15$$

2nd mode

$$\left. \begin{aligned} \frac{Y_0}{l_0 \theta_0} &= 1.15 \\ \frac{Y_1}{l_0 \theta_0} &= -0.85 \end{aligned} \right\} \frac{Y_1}{Y_0} = -0.74$$



$$\frac{Y_1}{Y_0} = \frac{-3.15}{-1.15} = 2.73$$

5-38

From Prob. 5-37 $\text{Det.} = \lambda^4 - 8\lambda^2 + 4$

$$\text{For Det} = 0, \quad \lambda = \frac{\omega}{\omega_h} = \begin{cases} 0.732 & 1^{\text{st}} \text{ mode} \\ 2.732 & 2^{\text{nd}} \text{ mode} \end{cases}$$

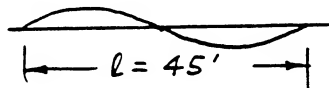
Using Cramer's rule

$$\frac{Y_0}{Y_G} = \frac{\begin{vmatrix} 1 & \lambda^2 \\ 1 & \frac{1}{3}(4-\lambda^2) \end{vmatrix}}{\lambda^4 - 8\lambda^2 + 4} = \frac{4}{3} \frac{(1-\lambda^2)}{\lambda^4 - 8\lambda^2 + 4}$$

$$\frac{\theta_0}{Y_G} = \frac{\begin{vmatrix} (1-\lambda^2) & 1 \\ 1 & 1 \end{vmatrix}}{\lambda^4 - 8\lambda^2 + 4} = \frac{-\lambda^2}{\lambda^4 - 8\lambda^2 + 4}$$

Plot for various values of λ to Check Fig P5-3B

5-39



Unfavorable speed is found

$$\text{from } v\gamma = l$$

$$v = \frac{l}{\gamma} = lf_m$$

From Prob 5-24

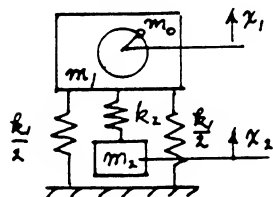
$$f_1 = 0.962 \quad \text{predominately bouncing}$$

$$f_2 = 1.327 \quad \text{" pitching}$$

$$v_1 = 45 \times 0.962 = 43.29 \text{ ft/s} = 29.5 \text{ mph}$$

$$v_2 = 45 \times 1.327 = 59.72 \text{ ft/s} = 40.7 \text{ mph}$$

5-40



$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = m_0 e \omega^2 \sin \omega t$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0$$

$$\frac{k_2}{m_2} \text{ should equal exciting freq } \omega^2$$

$$k_2 = \omega^2 m_2 = \left(\frac{2\pi 1800}{60} \right)^2 \left(\frac{50}{386} \right) = 4600 \text{ lb/in}$$

The absorber will then make $x_1 = 0$ and the absorber force will be equal and opposite to the exciting force.

$$k_2 x_2 = m_0 e \omega^2$$

$$x_2 = \frac{m_0 e \omega^2}{k_2} = \frac{m_0 e}{m_2} = \frac{2}{50} = 0.04 \text{ "}$$

5-41

$$m_1 \ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2(x_1 - x_2) = m_0 e \omega^2 (e^{i\omega t})$$

$$m_2 \ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

Let $x_1 = X_1 e^{i\omega t}$, $x_2 = X_2 e^{i\omega t}$ where X_1, X_2 are complex

$$\left(\frac{k_1+k_2}{m_1} - \omega^2 + i \frac{\omega c}{m_1}\right) X_1 - \left(\frac{k_2}{m_1} + \frac{i\omega c}{m_1}\right) X_2 = \frac{(m_0 e) \omega^2}{m_1}$$

$$-\left(\frac{k_2}{m_2} + \frac{i\omega c}{m_2}\right) X_1 + \left(\frac{k_2}{m_2} - \omega^2 + \frac{i\omega c}{m_2}\right) X_2 = 0$$

$$X_1 = \frac{(m_0 e) \omega^2 (k_2 - m_2 \omega^2 + i\omega c)}{[(k_1+k_2) - m_1 \omega^2 + i\omega c][k_2 - m_2 \omega^2 + i\omega c] - (k_2 + i\omega c)^2}$$

$$\frac{X_2}{X_1} = \frac{(k_2 - i\omega c)}{(k_1 - m_2 \omega^2 + i\omega c)}$$

5-42 $I \ddot{\theta}_1 + 4ka^2(\theta_1 - \theta_2) = M_0 \sin \omega t$

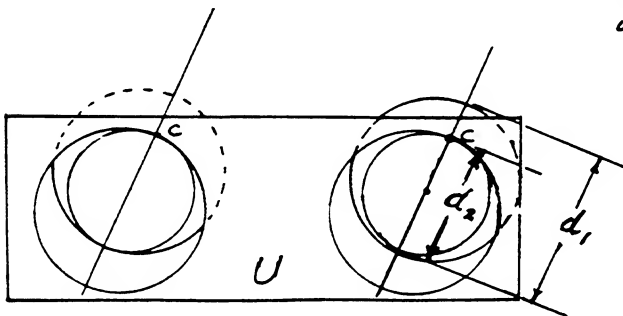
$$I_d \ddot{\theta}_2 + 4ka^2(\theta_2 - \theta_1) = 0$$

Let $\lambda = \frac{\omega^2 I}{ka^2}$ and $n = \frac{I_d}{I}$

$$\begin{bmatrix} (4-\lambda) & -1 \\ -1 & (4-n\lambda) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} M_0 \\ 0 \end{Bmatrix} \sin \omega t$$

$$\theta_1 = \frac{(4-n\lambda) M_0 \sin \omega t}{n\lambda^2 - 4(1+n)\lambda + 15} \quad \therefore \theta_1 = 0 \text{ when } \lambda = \frac{4}{n}$$

5-43 Point of contact c of U with small circle moves in circle of radius $r = d_1 - d_2$



5-44

From Eq. 5.7-5 the natural freq. is

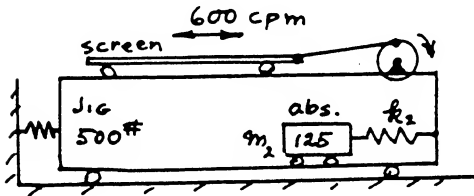
$$\omega_n = n \sqrt{\frac{R}{r}} = 4 (\text{rot. speed}) = 4n$$

$$\therefore \frac{r}{R} = \frac{1}{16} \quad r = d_1 - d_2 = \frac{3}{4}'' - d_2$$

$$\therefore r = \frac{R}{16} = \frac{4''}{16} = \frac{1}{4} = \frac{3}{4} - d_2 \quad d_2 = \frac{1}{2}''$$

5-45

$$f_1 = 400 \text{ cpm.}$$



$$\text{Excit. } \omega = \frac{2\pi 600}{60} = 20\pi \text{ rad/s.}$$

Nat. freq. of absorber must equal the excitation freq.

$$\omega_{22}^2 = \frac{k_2}{m_2} = \frac{386 k_2}{125} = (20\pi)^2$$

$$\therefore k_2 = 1278. \text{ \# / in}$$

Nat. freq. of system is found from the denominator of Eq 5.6-1 which can be reduced to

$$\left(\frac{\omega}{\omega_{22}}\right)^4 - \left[1 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 \left\{1 + \mu \left(\frac{\omega_{22}}{\omega_{11}}\right)^2\right\}\right] \left(\frac{\omega}{\omega_{22}}\right)^2 + \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 = 0$$

$$\mu = \frac{m_2}{m_1} = \frac{125}{500} = 0.25, \quad \left(\frac{\omega_{11}}{\omega_{22}}\right)^2 = \left(\frac{400}{600}\right)^2 = \frac{1}{2.25}, \quad \text{Let } \lambda = \frac{\omega}{\omega_{22}}$$

$$\lambda^4 - 1.695 \lambda^2 + \frac{1}{2.25} = 0, \quad \lambda^2 = \left(\frac{\omega}{\omega_{22}}\right)^2 = 0.845 \pm 0.164$$

5-46 Refer to Fig 5.6-3

With trial weight of 2 lb tuned to 232 rpm, the two nat. freqs. are

$$\left(\frac{\omega}{\omega_{22}}\right) = \frac{198}{232} = 0.854 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right) = \frac{272}{232} = 1.17$$

These numbers establish the mass ratio from Fig 5.6-3 to be $\mu \approx 0.10$

To move nat. freqs. outside specified freqs. of

$$\left(\frac{\omega}{\omega_{22}}\right) = \frac{160}{232} = 0.69 \quad \text{and} \quad \left(\frac{\omega}{\omega_{22}}\right) = \frac{320}{232} = 1.38$$

Fig 5.6-3 shows $\mu \geq 0.57$

5-46 Cont.

$$\text{Since } \mu_1 = \frac{(m_2)_1}{m_1} = \frac{2}{m_1} = 0.10 \quad \therefore m_1 = 20$$

$$\mu_2 = \frac{(m_2)_2}{m_1} = 0.57, \quad (m_2)_2 = .57 \times 20 = 11.4 \text{ lb.}$$

$$\text{The stiffness should be } k_2 = m_2 \omega^2 = \frac{11.4}{386} \left(\frac{2\pi \cdot 232}{60} \right)^2 = 17.9 \text{ \#/in}$$

5-47 Assume linear velocity distribution of fluid between disk and case. The torque is

$$\begin{aligned} T &= \mu (\text{velocity gradient})(\text{radius})(\text{area}) \\ &= 2 \int_{R_0}^R 2\pi \mu \left(\frac{\omega r}{h} \right) r^2 dr + 2\pi \mu \left(\frac{\omega R}{h} \right) R^2 b \\ &= 2\pi \mu \frac{\omega R^3}{h} \left[\frac{1}{2} \left(R - \frac{R_0^4}{R^3} \right) + b \right] \end{aligned}$$

5-48 Optimum damping given by Eq. 5.8-6

$$\zeta_o = \frac{\mu}{\sqrt{2(1+\mu)(2+\mu)}} = \frac{.25}{\sqrt{2(1.25)(2.25)}} = 0.1054$$

The freq. at which the damper is most effective (with peak ampl.) is given by Eq. 5.8-7

$$\frac{\omega}{\omega_n} = \sqrt{\frac{2}{2+\mu}} = \sqrt{\frac{2}{2.25}} = 0.943$$

5-49 The peak amplitude for any μ and ζ can be found from Fig 5.8-4. It is seen that the optimum (lowest point on curve) for $\mu = .25$ is $\zeta \approx .105$ as computed in Prob. 5-48. Thus

$$\begin{aligned} \frac{\theta_{\max} \zeta = .10}{\theta_{\max} \zeta = .105} &\approx 1.0 \end{aligned}$$

5-50 In Fig 5.8-3 all curves pass through a common point p . This point is at the frequency for optimum damping. Thus by equating $|\frac{K\theta_0}{M_0}|^2$ for $\zeta=0$ and $\zeta=\infty$ Eq. 5.8-7 is obtained

$$\frac{\mu^2 (\omega/\omega_n)^2}{\mu^2 (\omega/\omega_n)^2 (1 - \omega^2/\omega_n^2)} = \frac{4}{4 [\mu (\frac{\omega}{\omega_n})^2 - (1 - \omega^2/\omega_n^2)]^2}$$

or.
$$\frac{1}{1 - (\omega/\omega_n)^2} = \frac{1}{(\frac{\omega}{\omega_n})^2 (1 + \mu) - 1}$$

Thus
$$\omega/\omega_n = \sqrt{\frac{2}{2 + \mu}}$$

Eq. 5.8-6 is found by differentiating $|\frac{K\theta_0}{M_0}|^2$ w.r.t. $(\frac{\omega}{\omega_n})^2$, equating it to zero and substituting Eq. 5.8-7

Let $r^2 = \frac{2}{2 + \mu} = (\frac{\omega}{\omega_n})^2$ $(1 - r^2) = \frac{\mu}{2 + \mu} = r^2(1 + \mu) - 1$

$q = \frac{4\zeta^2}{\mu^2}$ Rewrite Eq. 5.8-5

$$|\frac{K\theta_0}{M_0}|^2 = \frac{r^2 + q}{r^2(1 - r^2)^2 + q[r^2(1 + \mu) - 1]^2}$$

$$\frac{\partial}{\partial r^2} = \frac{r^2(1 - r^2)^2 + q[r^2(1 + \mu) - 1]^2 - (r^2 + q)\{ (1 - r^2)^2 - 2r^2(1 - r^2) + 2q[r^2(1 + \mu) - 1](1 + \mu) \}}{[r^2(1 - r^2)^2 + q[r^2(1 + \mu) - 1]^2]^2} = 0$$

$$= r^2(1 - r^2)^2 + q(1 - r^2)^2 - (r^2 + q)\{ \text{same} \} = 0$$

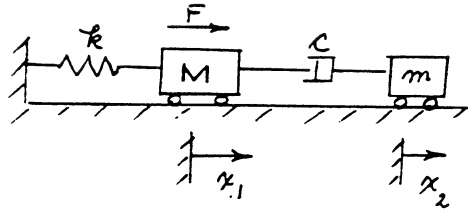
$$= (r^2 + q) \left[\left(\frac{\mu}{2 + \mu} \right)^2 - \left(\frac{\mu}{2 + \mu} \right)^2 + 2r^2 \left(\frac{\mu}{2 + \mu} \right) - 2q \left(\frac{\mu}{2 + \mu} \right) (1 + \mu) \right] = 0$$

$$\therefore r^2 - q(1 + \mu) = 0 \text{ and } q = \frac{r^2}{1 + \mu} = \frac{4\zeta^2}{\mu^2}$$

$$\zeta^2 = \frac{\mu^2}{4} \frac{1}{1 + \mu} \frac{2}{2 + \mu} = \frac{\mu^2}{2(1 + \mu)(2 + \mu)}$$

$$\zeta_{opt} = \frac{\mu}{\sqrt{2(1 + \mu)(2 + \mu)}}$$

5-51



$$M\ddot{x}_1 = -kx_1 - c(\dot{x}_1 - \dot{x}_2) + F e^{i\omega t}$$

$$m\ddot{x}_2 = c(\dot{x}_1 - \dot{x}_2) \quad \text{Let } x_1 = X_1 e^{i\omega t}, \quad x_2 = X_2 e^{i\omega t}$$

$$\left[\left(\frac{k}{M} - \omega^2 \right) + i \left(\frac{c\omega}{M} \right) \right] X_1 - i \left[\frac{c\omega}{M} \right] X_2 = \frac{F}{M}$$

$$\left[-\omega^2 + i \frac{c\omega}{m} \right] X_2 = i \left(\frac{c\omega}{m} \right) X_1 \quad \text{eliminate } X_2$$

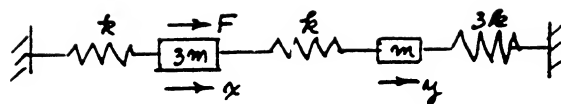
$$\left\{ \left[\left(\frac{k}{M} - \omega^2 \right) + i \left(\frac{c\omega}{M} \right) \right] - i \left(\frac{c\omega}{M} \right) \frac{i \left(\frac{c\omega}{m} \right)}{(-\omega^2 + i \frac{c\omega}{m})} \right\} X_1 = \frac{F}{M}$$

$$\left\{ \left[(k - M\omega^2) + ic\omega \right] \left(\frac{-m\omega^2 + ic\omega}{m} \right) + \frac{(c\omega)^2}{m} \right\} X_1 =$$

$$F \left(\frac{-m\omega^2 + ic\omega}{m} \right)$$

$$\therefore \frac{X_1}{F} = \frac{(\omega^2 m - ic\omega)}{m\omega^2(k - M\omega^2) + ic\omega[m\omega^2 - (k - M\omega^2)]}$$

5-52



$$\left. \begin{aligned} \ddot{x} &= -\frac{2}{3} \frac{k}{m} x + \frac{1}{3} \frac{k}{m} y + \frac{F}{3m} \\ \ddot{y} &= \frac{k}{m} x - 4 \frac{k}{m} y \end{aligned} \right\} \begin{aligned} \omega_1 &= 0.751 \sqrt{\frac{k}{m}} \\ \omega_2 &= 2.04 \sqrt{\frac{k}{m}} \end{aligned}$$

Let $k = m = 1$, then $\gamma_1 = 8.35 \text{ sec}$, $\gamma_2 = 3.07 \text{ sec}$.

Let $\Delta t = 0.20$

$$x_{i+1} = \ddot{x}_i \Delta t^2 + 2x_i - x_{i-1}$$

$$y_{i+1} = \ddot{y}_i \Delta t^2 + 2y_i - y_{i-1}$$

Initial Cond. ($I=1$)

$$x(1) = y(1) = \ddot{y}(1) = 0, \quad \ddot{x}_1(1) = \frac{100}{3}$$

From Eq. (4.7-8) $x(2) = \frac{1}{2} (0.20)^2 \left(\frac{100}{3} \right) = 0.666$

From Eq. (4.7-12) and DE for \ddot{y}

$$y(2) = \frac{1}{6} \Delta t^2 \ddot{y}(2) = \frac{\Delta t^2}{6} [x(2) - 4y(2)]$$

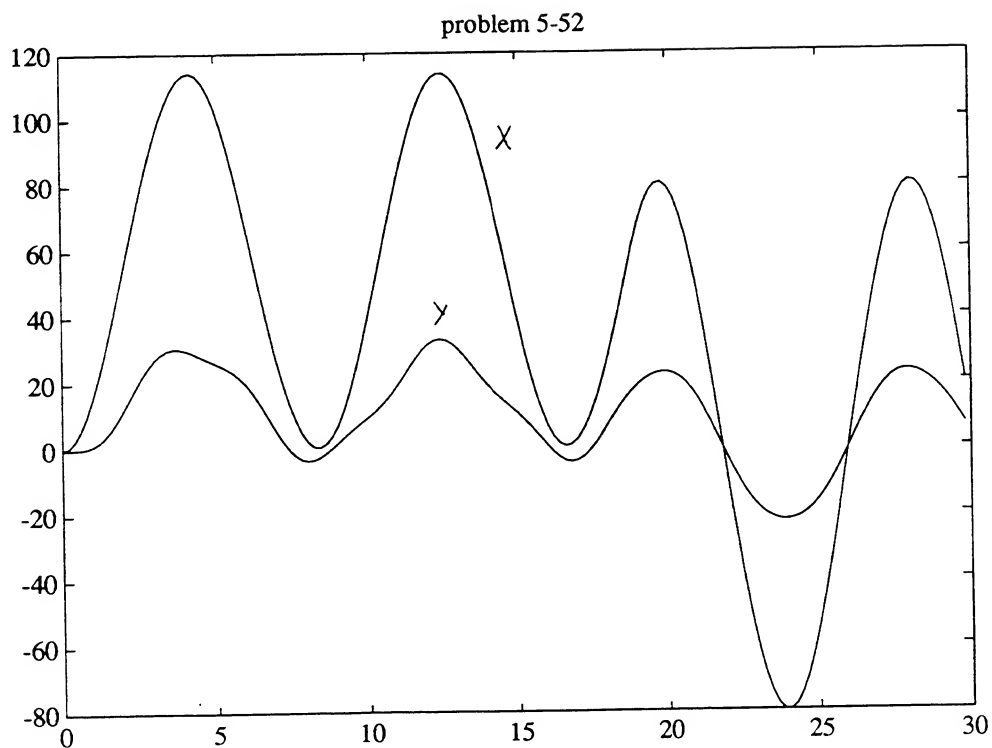
$$y(2) \left[1 + \frac{2}{3} \Delta t^2 \right] = \frac{1}{6} \Delta t^2 \times 0.666 \quad \therefore y(2) = 0.00432$$

computer program and plot of results follow.

5-52 cont

%This is the program for problem 5-52

```
clear; clg;
dt=0.20;
f=100
x(1)=0;
y(1)=0;
dx(1)=0.0;
dx(1)=f/3;
t(1)=0;
x(2)=0.5*dx(1)*dt^2;
y(2)=(x(2)*dt^2)/(6+4*dt^2);
dx(2)=(-2*x(2)+y(2)+f)/3;
dy(2)=x(2)-4*y(2);
t(2)=dt
for i=3:150
    if i<95
        f=100;
    else
        f=0;
    end
    t(i)=(i-1)*dt;
    x(i)=dx(i-1)*dt^2+2*x(i-1)-x(i-2);
    y(i)=dy(i-1)*dt^2+2*y(i-1)-y(i-2);
    dx(i)=(-2*x(i)+y(i)+f)/3;
    dy(i)=x(i)-4*y(i);
end
```



5-53

$$100 \ddot{x} = -4 \times 10^3 (x - y)$$

$$100 \ddot{y} = 4 \times 10^3 (x - y) - 6 \times 10^3 (y - z)$$

$$\therefore \ddot{x} = -40x + 40y$$

$$\ddot{y} = 40x - 100y + 60z$$

$$z = 10 \sin \pi t$$

Use Eq. 4.7-12 to start computation

$$x(z) = \frac{1}{6} \ddot{x}(z) h^2 = \frac{h^2}{6} [-40x(z) + 40y(z)]$$

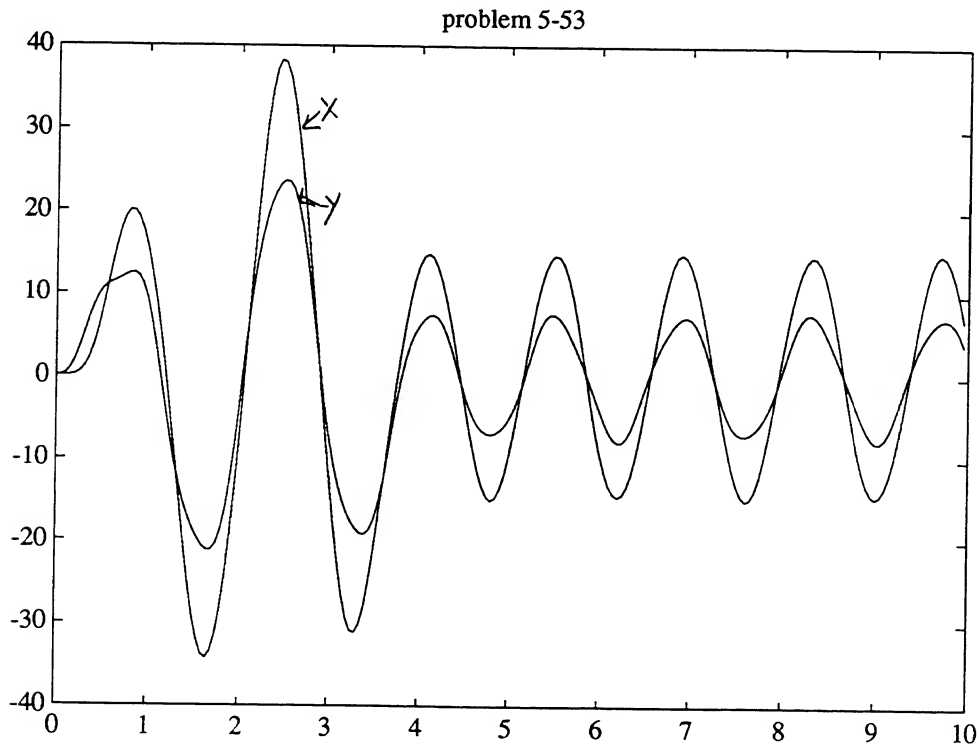
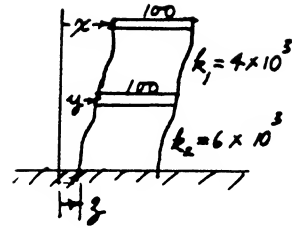
$$y(z) = \frac{h^2}{6} \ddot{y}(z) = \frac{h^2}{6} [40x(z) - 100y(z) + 60z(z)]$$

$$x(z) \left[1 + \frac{40}{6} h^2 \right] = y(z) \frac{40}{6} h^2$$

$$y(z) \left[1 + \frac{100}{6} h^2 \right] = x(z) \frac{40}{6} h^2 + 10 h^2 z(z)$$

$$\therefore x(z) = \frac{40}{6} h^2 \frac{1}{1 + \frac{40}{6} h^2} \cdot \frac{1}{1 + \frac{100}{6} h^2} \cdot \left[x(z) \frac{40}{6} h^2 + 10 h^2 z(z) \right]$$

$$\left. \begin{aligned} x(z) &= 2400 h^4 z(z) / (36 + 840 h^2 + 2400 h^4) \\ y(z) &= (1 + \frac{40}{6} h^2) x(z) / \frac{40}{6} h^2 \end{aligned} \right\} \text{starting equations}$$



5-53 CONT.

```
%This is the matlab code for problem 5-53
clear; clg;
dt=0.05;
f=100;
x(1)=0.0;
y(1)=0.0;
dy(1)=0.0;
dx(1)=0.0;
t(1)=0;
z(1)=0;
t(2)=dt;
z(2)=10*sin(pi*dt);
x(2)=(400*6*dt^4*z(2))/(36+840*dt^2+2400*dt^4);
y(2)=(1+40/6*dt^2)*x(2)/(40/6*dt^2);
dx(2)=-40*x(2)+40*y(2);
dy(2)=40*x(2)-100*y(2)+60*z(2);
for i=3:200
    t(i)=(i-1)*dt;
    if t(i)<4
        z(i)=10*sin(pi*t(i));
    else
        z(i)=0;
    end
    x(i)=dx(i-1)*dt^2+2*x(i-1)-x(i-2);
    y(i)=dy(i-1)*dt^2+2*y(i-1)-y(i-2);
    dx(i)=-40*x(i)+40*y(i);
    dy(i)=40*x(i)-100*y(i)+60*z(i);
end
```

5-54

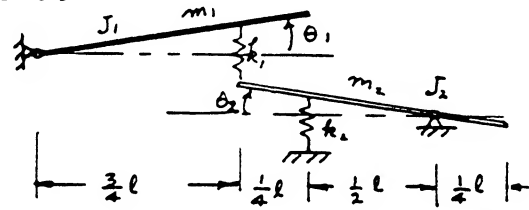
$$\left. \begin{aligned} J_1 \ddot{\theta}_1 &= K_1(\theta_2 - \theta_1) \\ J_2 \ddot{\theta}_2 &= -K_1(\theta_2 - \theta_1) + K_2(\theta_3 - \theta_2) \\ J_3 \ddot{\theta}_3 &= -K_2(\theta_3 - \theta_2) \end{aligned} \right\} \begin{aligned} \text{If } \phi &= (\theta_2 - \theta_1) \\ \psi &= (\theta_3 - \theta_2) \\ \text{the DEs. can be} \\ &\text{written as} \end{aligned}$$

$$\left. \begin{aligned} J_2 \ddot{\phi} &= -K_1 \left(1 + \frac{J_2}{J_1}\right) \phi + K_2 \psi \\ J_3 \ddot{\psi} &= K_1 \frac{J_3}{J_2} \phi - K_2 \left(1 + \frac{J_3}{J_2}\right) \psi \end{aligned} \right\} \begin{aligned} \text{DEs in two} \\ \text{coordinates} \end{aligned}$$

Freq. Eq. from either set of equations is

$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_2} \left(1 + \frac{K_1}{K_2} + \frac{J_2}{J_3}\right) \right] \omega^2 + \frac{K_1 K_2}{J_1 J_2} \left(\frac{J_1 + J_2 + J_3}{J_3} \right) = 0$$

5-55



$$\begin{aligned} J_1 \ddot{\theta}_1 &= -\frac{3}{4} l k_1 (\theta_1 - \theta_2) \frac{3}{4} l \\ J_2 \ddot{\theta}_2 &= \frac{3}{4} l k_1 (\theta_1 - \theta_2) \frac{3}{4} l - \frac{1}{2} l k_2 \theta_2 \frac{1}{2} l \end{aligned}$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + l^2 k_1 \begin{bmatrix} \frac{9}{16} & -\frac{9}{16} \\ -\frac{9}{16} & (\frac{9}{16} + \frac{1}{4} \frac{k_2}{k_1}) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$J_1 = m_1 l^2 \frac{1}{3} \quad J_2 = m_2 l^2 \left(\frac{1}{12} + \frac{1}{16} \right)$$

5-56

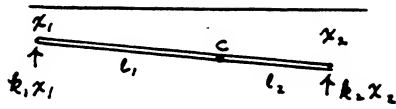
$$\begin{bmatrix} [K_1(1 + \frac{J_2}{J_1}) - \omega^2 J_1] & -K_2 \\ -K_1 \frac{J_3}{J_2} & [K_2(1 + \frac{J_3}{J_2}) - \omega^2 J_3] \end{bmatrix} \begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{\phi}{\psi} = \frac{K_2}{K_1(1 + \frac{J_2}{J_1}) - \omega^2 J_1} = \frac{\theta_2 - \theta_1}{\theta_3 - \theta_2} = \frac{\frac{\theta_2}{\theta_1} - 1}{\frac{\theta_3}{\theta_1} - \frac{\theta_2}{\theta_1}}$$

Assign numerical values for K_1, K_2, J_1, J_2, J_3 and prove

$$J_1 \theta_1^{(i)} \theta_1^{(j)} + J_2 \theta_2^{(i)} \theta_2^{(j)} + J_3 \theta_3^{(i)} \theta_3^{(j)} = 0$$

5-57



$$\begin{aligned} \text{Displ. of } c &= x_1 + \frac{l_1}{l_1 + l_2} (x_2 - x_1) \\ &= \frac{l_2 x_1 + l_1 x_2}{l_1 + l_2} \end{aligned}$$

$$T = \frac{1}{2} m \left(\frac{l_2 \dot{x}_1 + l_1 \dot{x}_2}{l_1 + l_2} \right)^2 + \frac{1}{2} J_c \left(\frac{\dot{x}_2 - \dot{x}_1}{l_1 + l_2} \right)^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 \quad \therefore \text{Dynamic coupling exists}$$

5-58. Equations to be solved are:

$$\ddot{x} = -720x + 360y \quad (\text{see sec. 5.5})$$

$$\ddot{y} = 1440(x - y) + 160$$

$$(s^2 + 720)\bar{x}(s) - 360\bar{y}(s) = 0$$

$$-1440\bar{x}(s) + (s^2 + 1440)\bar{y}(s) = \frac{160}{s}$$

$$\text{from 1st eq. } \bar{x}(s) = \left(\frac{360}{s^2 + 720} \right) \bar{y}(s)$$

$$\text{subst. into 2nd eq. } -1440 \left(\frac{360}{s^2 + 720} \right) \bar{y}(s) + (s^2 + 1440)\bar{y}(s) = \frac{160}{s}$$

x by $(s^2 + 720)$

$$[-518,400 + s^4 + 2160s^2 + 1,036,800]\bar{y}(s) = \frac{160(s^2 + 720)}{s}$$

$$[s^4 + 2160s^2 + 518,400]\bar{y}(s) = \frac{160}{s}(s^2 + 720)$$

$$\bar{y}(s) = \frac{160(s^2 + 720)}{s(s^4 + 2160s^2 + 518,400)}$$

$$\text{Roots: } s^2 = -1080 \pm \sqrt{1,166,400 - 518,400}$$

$$= -1080 \pm \sqrt{648,000}$$

$$= -1080 \pm 804.9845$$

also $s = 0$ is a root.

5-58 Cont.

$$\therefore S_1^2 = -275.0155 \quad S_1 = \pm i 16.5836$$

$$S_2^2 = -1884.9845 \quad S_2 = \pm i 43.4164$$

Rewrite $\bar{y}(s)$ as

$$\begin{aligned} \bar{y}(s) &= \frac{160 (s^2 + 720)}{s (s^2 + 275.0155) (s^2 + 1884.9845)} \\ &= \frac{160 (s^2 + 720)}{s (s + i 16.5836) (s - i 16.5836) (s + i 43.4164) (s - i 43.4164)} \\ &= \frac{C_1}{(s + i 16.5836)} + \frac{C_2}{(s - i 16.5836)} + \frac{C_3}{(s + i 43.4164)} \\ &\quad + \frac{C_4}{(s - i 43.4164)} + \frac{C_5}{s} \end{aligned}$$

$$\begin{aligned} C_1 &= \lim_{s \rightarrow S_1} \left[\frac{160 (s^2 + 720) (s - i 16.5836)}{s (s + i 16.5836) (s - i 16.5836) (s^2 + 1884.9845)} \right] \\ &= \frac{160 (-275.0155 + 720)}{(-i 16.5836) (-i 33.1672) (-275.0155 + 1884.9845)} = -.0804007 \end{aligned}$$

$$C_2 = \lim_{s \rightarrow S_1} \left[\frac{160 (s^2 + 720) (s - i 16.5836)}{s (s + i 16.5836) (s - i 16.5836) (s^2 + 1884.98)} \right] = -.0804007$$

$$C_3 = \lim_{s \rightarrow S_2} \left[\frac{160 (s^2 + 720) (s + i 43.4164)}{s (s^2 + 275.0155) (s + i 43.4164) (s - i 43.4164)} \right] = -.03071$$

$$C_4 = C_3$$

$$C_5 = \frac{160 (720)}{(275.0155) (1884.9845)} = 0.2222$$

5-58 Cont.

$$\bar{y}(s) = \frac{.2222}{s} - \frac{.0804007}{(s+i16.5836)} - \frac{.0804007}{(s-i16.5836)} \\ - \frac{.03071}{(s+i43.4164)} - \frac{.03071}{(s-i43.4164)}$$

Inverse

$$y(t) = 0.2222 - .0804007 \left(e^{-i16.5836t} + e^{i16.5836t} \right) \\ - .03071 \left(e^{-i43.4164t} + e^{i43.4164t} \right)$$

$$= 0.2222 - .1608 \cos 16.58t - .06142 \cos 43.41t$$

$$= .1608 (1 - \cos 16.58t) + .06142 (1 - \cos 43.41t) \text{ meters}$$

move dec. pt. 2 places to left for centimeters

Solution for $x(t)$

$$\bar{x}(s) = \frac{360}{(s^2 + 720)} \quad \bar{y}(s) = \frac{360}{(s^2 + 720)} \cdot \frac{160(s^2 + 720)}{s(s^2 + 275.0155)(s^2 + 1884.9845)}$$

\therefore roots are same. Solve as above with partial fractions, the equation

$$\bar{x}(s) = \frac{57600}{s(s^2 + 275.0155)(s^2 + 1884.9845)}$$

Ans.

$$x(t) = .1301(1 - \cos 16.58t) - .0190(1 - \cos 43.41t) \text{ meters.}$$

5-59 Examine for example the subsidiary sol. for Prob 5-58

$$\bar{y}(s) = \sum \frac{C_i}{s - s_i} \quad \text{where } s_i \text{ are roots}$$

\therefore Results are sums of normal modes

$$m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \sin \omega t$$

$$ms^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{x}_1(s) \\ \bar{x}_2(s) \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} \bar{x}_1(s) \\ \bar{x}_2(s) \end{Bmatrix} = \begin{Bmatrix} \frac{F_0 \omega}{s^2 + \omega^2} + msx_1(0) + m\dot{x}_1(0) \\ msx_2(0) + m\dot{x}_2(0) \end{Bmatrix}$$

$$\begin{bmatrix} (ms^2 + 2k) & -k \\ -k & (ms^2 + 2k) \end{bmatrix} \begin{Bmatrix} \bar{x}_1(s) \\ \bar{x}_2(s) \end{Bmatrix} = \begin{Bmatrix} \text{''} \\ \text{''} \end{Bmatrix}$$

$$\text{Det. of matrix } (ms^2 + 2k)^2 - k^2 = 0, \quad s^2 = -\frac{2k}{m} \pm \frac{k}{m}$$

$$\therefore \left(s^2 + \frac{k}{m}\right) \left(s^2 + \frac{3k}{m}\right) m$$

$$\bar{x}_1(s) = \frac{\begin{vmatrix} \left[\frac{F_0 \omega}{s^2 + \omega^2} + msx_1(0) + m\dot{x}_1(0)\right] - k \\ [msx_2(0) + m\dot{x}_2(0)] \end{vmatrix} \cdot (ms^2 + 2k)}{\left(s^2 + \frac{k}{m}\right) \left(s^2 + \frac{3k}{m}\right) m}$$

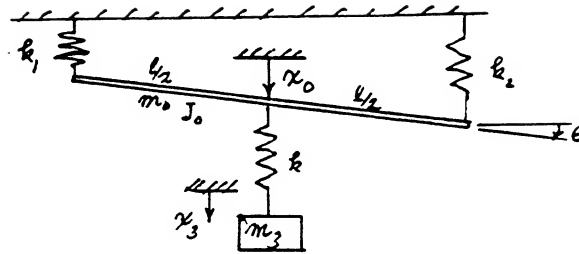
$$\bar{x}_2(s) = \frac{\begin{vmatrix} (ms^2 + 2k) \left[\frac{F_0 \omega}{s^2 + \omega^2} + msx_1(0) + m\dot{x}_1(0)\right] \\ -k [msx_2(0) + m\dot{x}_2(0)] \end{vmatrix}}{\left(s^2 + \frac{k}{m}\right) \left(s^2 + \frac{3k}{m}\right) m}$$

For steady state vibration $x(0), \dot{x}(0) = 0$ and $s = i\omega$

$$x_1(t) = \frac{\left(2\frac{k}{m} - \omega^2\right) F_0 \sin \omega t}{\left(\frac{k}{m} - \omega^2\right) \left(3\frac{k}{m} - \omega^2\right)}$$

$$x_2(t) = \frac{k F_0 \sin \omega t}{\left(\frac{k}{m} - \omega^2\right) \left(3\frac{k}{m} - \omega^2\right) m}$$

5-61



$$m_0 \ddot{x}_0 = -k_1(x_0 - \frac{l_2}{2}\theta) - k_2(x_0 + \frac{l_2}{2}\theta) + k_3(x_3 - x_0)$$

$$J_0 \ddot{\theta} = k_1(x_0 - \frac{l_2}{2}\theta) \frac{l_2}{2} - k_2(x_0 + \frac{l_2}{2}\theta) \frac{l_2}{2}$$

$$m_3 \ddot{x}_3 = -k_3(x_3 - x_0)$$

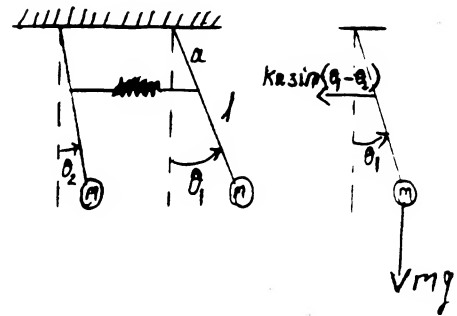
$$\begin{bmatrix} m_0 & & \\ & J_0 & \\ & & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_0 \\ \ddot{\theta} \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2+k_3) & (k_2-k_1)\frac{l_2}{2} & -k_3 \\ (k_2-k_1)\frac{l_2}{2} & (k_1+k_2)(\frac{l_2}{2})^2 & 0 \\ -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_0 \\ \theta \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

5-62

$$\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2+k_5) & -k_2 & -k_5 \\ -k_2 & (k_2+k_3+k_4) & -k_3 \\ -k_5 & -k_3 & (k_3+k_5) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

5-63

Not assuming small oscillations, we have



$$m l^2 \ddot{\theta}_1 = -m g l \sin \theta_1 - k a^2 \sin(\theta_1 - \theta_2)$$

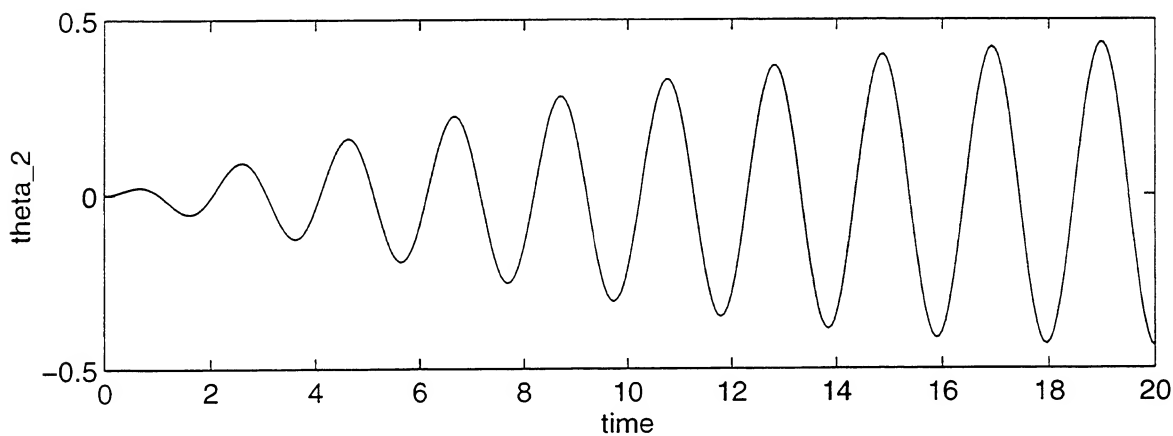
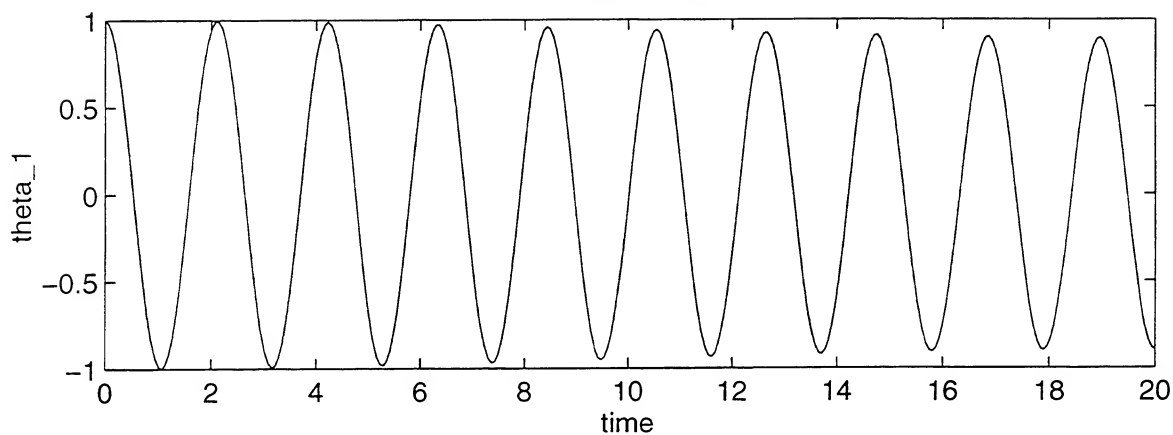
$$m l^2 \ddot{\theta}_2 = -m g l \sin \theta_2 + k a^2 \sin(\theta_1 - \theta_2)$$

```
clear
global m l a k g
'Problem 5-63 Solution'
m=input('Enter the value of the mass (kg) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
l=input('Enter the value of the pendulum length (m) : ')
a=input('Enter the value of the distance a (m): ')
x0=input('Enter the four dimensional initial condition [x1 x1d x2 x2d] : ')
tf=input('Enter the value of the final time (s) : ')
t0=0;g=9.81;
[t,x]=ode45('p563d',t0,tf,x0);
subplot(2,1,1);plot(t,x(:,1));xlabel('time');ylabel('theta_1');title('Problem 5-63')
subplot(2,1,2);plot(t,x(:,3));xlabel('time');ylabel('theta_2');
```

```
function xdot=p563d(t,x)
global m l a k g
xdot=[x(2);
      -(g/l)*sin(x(1))-(k/m)*(a/l)^2*sin(x(1)-x(3));
      x(4);
      -(g/l)*sin(x(3))+(k/m)*(a/l)^2*sin(x(1)-x(3))];
```

$$m=k=l=1, \quad a=0.5, \quad x_0 = [1 \ 0 \ 0 \ 0]$$

Problem 5-63



5-64

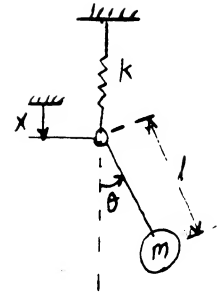
$$f_i(x_1, x_2) = f_i(x_1 - x_1, x_2 - x_1) = f_i(0, x_2 - x_1), \quad i=1, 2$$

Let $y = x_2 - x_1$. Therefore

$$\begin{aligned} \ddot{y} &= \ddot{x}_2 - \ddot{x}_1 = f_2(x_1, x_2) - f_1(x_1, x_2) \\ &= f_2(0, x_2 - x_1) - f_1(0, x_2 - x_1) \\ &= f_2(0, y) - f_1(0, y) = f(0, y). \end{aligned}$$

5-65

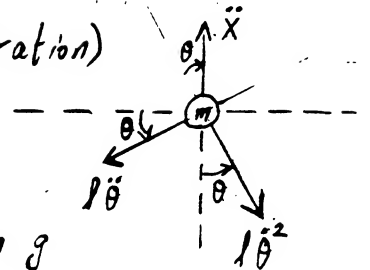
Since the rod that connects the spring and the mass m is massless, the spring force is transmitted from one end of the rod to the other. Thus, in the vertical direction, we have



$$m a_v = -kx \quad (a_v \text{ is the vertical acceleration})$$

$$m(\ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta) = -kx$$

$$\therefore \ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta + \frac{k}{m}x = 0$$



Note that the gravitational acceleration g was not included since we are measuring x from the equilibrium position.

In the tangential direction, we have

$$m a_t = -mg\sin\theta \quad (a_t \text{ is the tangential acceleration})$$

$$m(l\ddot{\theta} - \ddot{x}\sin\theta) = -mg\sin\theta$$

$$\therefore l\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0$$

The equations of motion are

$$\ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta + (k/m)x = 0$$

$$l\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0$$

To find the natural frequencies, we assume $x = X\sin\omega t$ and $\theta = \Theta\sin\omega t$ (*)
It is clear that $\Theta=0$, $\omega = \omega_n = \sqrt{k/m}$ is a solution which represents the vertical oscillations (no rotation).

Thus, $\left(\frac{\theta}{X}\right)_1 = 0$ and $\omega_1 = \sqrt{\frac{k}{m}}$

In this case, we have $\ddot{X} + \frac{k}{m}X = 0$.

Now, assuming small θ ($\sin\theta \approx \theta$ and $\cos\theta \approx 1$) the equations of motion become

$$\begin{aligned}\ddot{X} - l\ddot{\theta} - l\dot{\theta}^2 + \omega_n^2 X &= 0 \\ l\ddot{\theta} - \ddot{X}\theta + g\theta &= 0\end{aligned}$$

Plugging in (*) in the second equation of motion, we get

$$\begin{aligned}-\omega^2 l \theta \sin \omega t + \omega^2 X \theta \sin^2 \omega t + g \theta \sin \omega t &= 0 \\ \therefore -1 + \left(\frac{X}{l}\right) \sin \omega t + \left(\frac{\omega_n^2}{\omega}\right)^2 &= 0, \text{ where } \omega_n^2 = g/l\end{aligned}$$

if $X \ll l$ then $\left(\frac{X}{l}\right) \sin \omega t \ll 1$. Hence

$$\frac{\omega_n^2}{\omega} = 1 \Rightarrow \omega = \omega_n = \sqrt{g/l} \text{ which is the same as the frequency of a simple pendulum}$$

The above approximations lead to pure rotations. In this case, $\left(\frac{\theta}{X}\right)_2 = \infty$ (very large) and $\omega_2 = \sqrt{g/l}$.

To get a better result one must plug in (*) in both differential equations and solve for ω_2 .

large $k \Rightarrow$ large ω_1 and vice versa.
 \therefore In order for the skiers not to experience very fast oscillations k should be low.

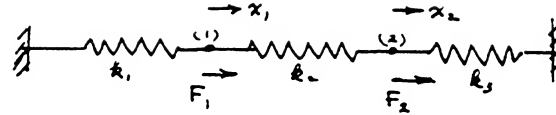
6-1

$$F_1 = -k_1 x_1 - k_2 (x_1 - x_2)$$

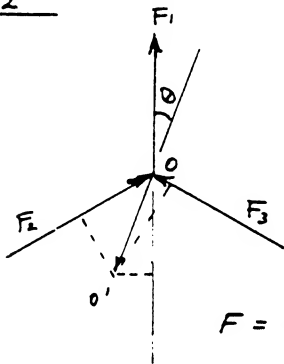
$$F_2 = -k_3 x_2 - k_2 (x_2 - x_1)$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [K]^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{-1}{k_1 k_2 + k_1 k_3 + k_2 k_3} \begin{bmatrix} (k_2+k_3) & k_2 \\ k_2 & (k_1+k_2) \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



6-2



Let O move to O' with δ small

$$\text{Then } F_1 = \delta k \cos \theta$$

$$F_2 = \delta k \cos(60 - \theta)$$

$$F_3 = \delta k \cos(60 + \theta)$$

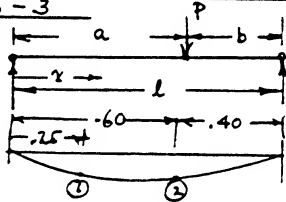
Force along δ is

$$F = F_1 \cos \theta + F_2 \cos(60 - \theta) + F_3 \cos(60 + \theta)$$

$$= k \delta [\cos^2 \theta + \cos^2(60 - \theta) + \cos^2(60 + \theta)] = 1.5 k \delta$$

$$\therefore \frac{\delta}{F} = \frac{1}{1.5k} \text{ and independent of } \theta$$

6-3



$$y = \frac{Pbx}{6EI \cdot l} (l^2 - x^2 - b^2) \quad x \geq a$$

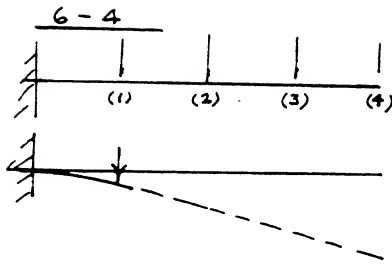
$$\text{Position ①} = 0.25l, \quad \text{Position ②} = 0.60l$$

$$a_{11} = \frac{(0.75)(0.25)l^3}{6EI} (1 - 0.25^2 - 0.75^2) = 0.0114 \frac{l^3}{EI}$$

$$a_{12} = a_{21} = \frac{(0.15)(0.40)l^3}{6EI} (1 - 0.40^2 - 0.25^2) = 0.0130 \frac{l^3}{EI}$$

$$a_{22} = \frac{(0.40)(0.60)l^3}{6EI} (1 - 0.60^2 - 0.40^2) = 0.0192 \frac{l^3}{EI}$$

$$a = \frac{l^3}{EI} \begin{bmatrix} 0.0114 & 0.0130 \\ 0.0130 & 0.0192 \end{bmatrix}$$



Place unit load at
① and repeat

$$a_{11} = \frac{\left(\frac{l}{4}\right)^3}{3EI} = \frac{l^3}{192EI}$$

$$\theta_{11} = \frac{\left(\frac{l}{4}\right)^2}{2EI} = \frac{l^2}{32EI}$$

$$a_{21} = a_{11} + \theta_{11} \frac{l}{4} = \frac{l^3}{192EI} + \frac{l^3}{128EI} = \frac{2.50 l^3}{192EI}$$

$$a_{31} = a_{21} + \frac{l^3}{128EI} = \frac{4 l^3}{192EI}$$

$$a_{41} = a_{31} + \frac{l^3}{128EI} = \frac{5.50 l^3}{192EI}$$

$$a_{22} = \frac{\left(\frac{l}{2}\right)^3}{3EI} = \frac{l^3}{24EI}$$

$$\theta_{22} = \frac{\left(\frac{l}{2}\right)^2}{2EI} = \frac{l^2}{8EI}$$

$$a_{32} = a_{22} + \theta_{22} \frac{l}{4} = \left(\frac{1}{24} + \frac{1}{32}\right) \frac{l^3}{EI} = \frac{7}{96} \frac{l^3}{EI}$$

$$a_{42} = \left(\frac{7}{96} + \frac{1}{32}\right) \frac{l^3}{EI} = \frac{10}{96} \frac{l^3}{EI}$$

$$a_{33} = \frac{\left(\frac{3l}{4}\right)^3}{3EI} = \frac{9}{64} \frac{l^3}{EI}$$

$$\theta_{33} = \frac{\left(\frac{3l}{4}\right)^2}{2EI} = \frac{9}{32} \frac{l^2}{EI}$$

$$a_{43} = a_{33} + \theta_{33} \frac{l}{4} = \left(\frac{9}{64} + \frac{9}{128}\right) \frac{l^3}{EI} = \frac{27}{128} \frac{l^3}{EI}$$

$$a_{44} = \frac{l^3}{3EI}$$

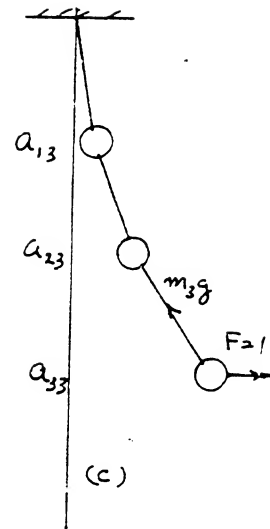
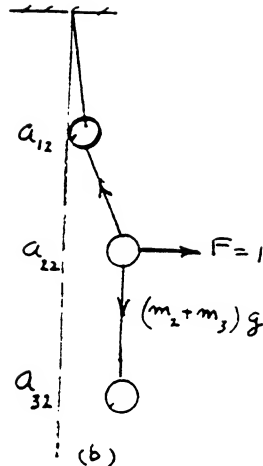
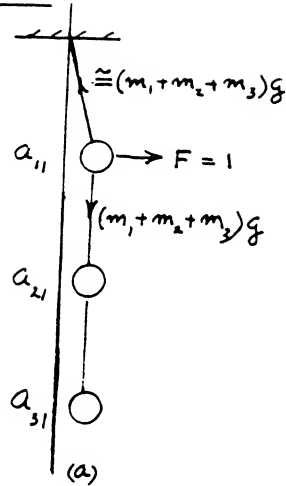
$$a = \frac{l^3}{EI} \begin{bmatrix} \frac{1}{192} & \frac{2.5}{192} & \frac{4}{192} & \frac{5.5}{192} \\ \frac{2.5}{192} & \frac{1}{24} & \frac{7}{96} & \frac{10}{96} \\ \frac{4}{192} & \frac{7}{96} & \frac{9}{64} & \frac{27}{128} \\ \frac{5.5}{192} & \frac{10}{96} & \frac{27}{128} & \frac{1}{3} \end{bmatrix} = \frac{l^3}{192EI} \begin{bmatrix} 1 & 2.5 & 4.0 & 5.5 \\ 2.5 & 8.0 & 14.0 & 20.0 \\ 4.0 & 14.0 & 27.0 & 40.5 \\ 5.5 & 20.0 & 40.5 & 64.0 \end{bmatrix}$$

EDU>inv(a)

ans =

6.2680	-3.9381	1.4845	-0.2474
-3.9381	4.7835	-3.1959	0.8660
1.4845	-3.1959	3.2990	-1.2165
-0.2474	0.8660	-1.2165	0.5361

6-5



From (a)

$$\frac{a_{11}}{l_1} (m_1 + m_2 + m_3) g = 1$$

$$a_{11} = a_{21} = a_{31} = \frac{1}{m_1 + m_2 + m_3} \cdot \frac{l_1}{g}$$

From (b)

$$\left(\frac{a_{22} - a_{12}}{l_2} \right) (m_2 + m_3) g = 1$$

$$a_{22} = \frac{1}{m_2 + m_3} \cdot \frac{l_2}{g} + \frac{1}{m_1 + m_2 + m_3} \cdot \frac{l_1}{g} = a_{32}$$

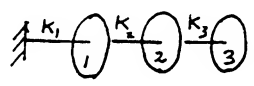
From (c)

$$\left(\frac{a_{33} - a_{23}}{l_3} \right) m_3 g = 1$$

$$a_{33} = \frac{1}{m_3} \cdot \frac{l_3}{g} + \frac{1}{m_2 + m_3} \cdot \frac{l_2}{g} + \frac{1}{m_1 + m_2 + m_3} \cdot \frac{l_1}{g}$$

$$a_{ij} = a_{ji}$$

6-6



Give each disk a unit rotation holding other disks with zero rotation. Torque required is then:

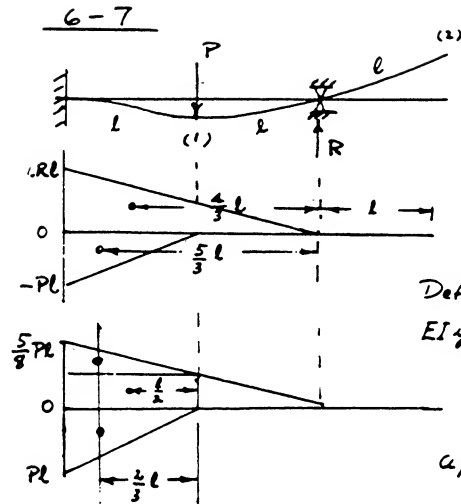
$$\theta_1 = 1.0 \quad T_1 = (K_1 + K_2), \quad T_2 = -K_1, \quad T_3 = 0$$

$$\theta_2 = 1.0 \quad T_1 = -K_2, \quad T_2 = (K_2 + K_3), \quad T_3 = -K_3$$

$$\theta_3 = 1.0 \quad T_1 = 0, \quad T_2 = -K_3, \quad T_3 = K_3$$

$$[K] = \begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \quad [a] = \begin{bmatrix} \frac{1}{K_1} & \frac{1}{K_1} & \frac{1}{K_1} \\ \frac{1}{K_1} & (\frac{1}{K_1} + \frac{1}{K_2}) & (\frac{1}{K_1} + \frac{1}{K_2}) \\ \frac{1}{K_1} & (\frac{1}{K_1} + \frac{1}{K_2}) & (\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}) \end{bmatrix}$$

For flexibility apply unit torque to each disk in turn and measure rotation.



Defl. at $R = 0$

$$\frac{1}{2} (2Rl \times 2l) \frac{4}{3}l - \frac{1}{2} (Pl \times l) \frac{5}{3}l = 0$$

$$\therefore R = \frac{5}{16} P$$

$$2Rl = \frac{5}{8} Pl$$

Defl. at (1) = moment of area about (1)

$$EI y_1 = \left(\frac{5}{16} Pl \times l \right) \frac{l}{2} + \frac{1}{2} \left(\frac{5}{16} Pl^2 \right) \frac{2}{3}l - \frac{1}{2} (Pl^2) \frac{2}{3}l$$

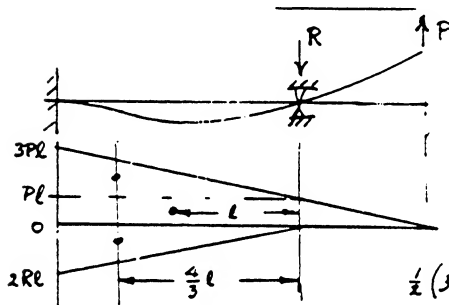
$$= Pl^3 \left(\frac{5}{32} + \frac{5}{48} - \frac{1}{3} \right) = -\frac{3.5}{48} Pl^3$$

$$a_{11} = \frac{3.50}{48} \frac{l^3}{EI} = \frac{7}{96} \frac{l^3}{EI}$$

Defl. at (2) = moment of area about (2)

$$EI y_2 = \frac{1}{2} \left[\frac{5}{8} Pl \times 2l \right] \frac{7}{3}l - \frac{1}{2} [Pl^2] \frac{8}{3}l = Pl^3 \left(\frac{35}{24} - \frac{8}{6} \right) = \frac{1}{8} Pl^3$$

$$\therefore a_{21} = a_{12} = \frac{1}{8} \frac{l^3}{EI}$$



Defl. at $R = 0$

$$(2Pl^2)l + \frac{1}{2} (2Pl \times 2l) \frac{4}{3}l - \frac{1}{2} (2Rl \times 2l) \frac{4}{3}l = 0$$

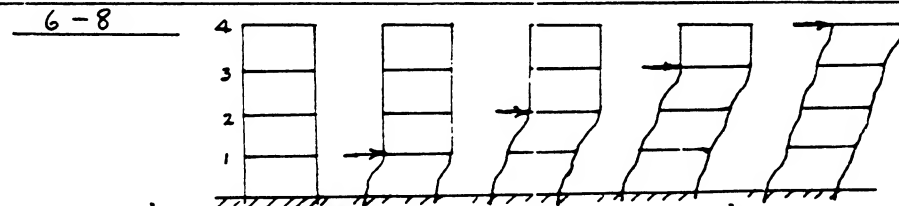
$$Pl^3 \left(2 + \frac{8}{3} \right) = Rl^3 \left(\frac{8}{3} \right) \therefore R = \frac{7}{4} P$$

For a_{22}

$$\frac{1}{2} (3Pl \times 3l) 2l - \frac{1}{2} \left(\frac{7}{2} Pl \times 2l \right) \left(l + \frac{4}{3}l \right) = EI y_2$$

$$Pl^3 (9 - 8.666) = \left(\frac{5.7}{6} - \frac{4.9}{6} \right) Pl^3 = \frac{5}{6} Pl^3 \therefore a_{22} = \frac{5}{6} \frac{l^3}{EI}$$

$$[a] = \frac{l^3}{EI} \begin{bmatrix} 7/96 & 1/8 \\ 1/8 & 5/6 \end{bmatrix}$$



$$a_{11} = \frac{1}{12} \frac{l^3}{EI}$$

$$a_{12} = \frac{1}{12} \frac{l^3}{EI}$$

$$a_{13} = \frac{1}{12} \frac{l^3}{EI}$$

$$a_{14} = \frac{1}{12} \frac{l^3}{EI}$$

$$a_{21} = \frac{1}{12} "$$

$$a_{22} = \frac{2}{12} "$$

$$a_{23} = \frac{2}{12} "$$

$$a_{24} = \frac{2}{12} "$$

$$a_{31} = \frac{1}{12} "$$

$$a_{32} = \frac{2}{12} "$$

$$a_{33} = \frac{3}{12} "$$

$$a_{34} = \frac{3}{12} "$$

$$a_{41} = \frac{1}{12} "$$

$$a_{42} = \frac{2}{12} "$$

$$a_{43} = \frac{3}{12} "$$

$$a_{44} = \frac{4}{12} "$$

$$[a] = \frac{l^3}{12EI} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

6-8 Cont.

$$[k] = [a]^{-1}$$

$$D = \det. [a] = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 2(12-9) - 1(12-9) + 1(8-6) - 1(0) - 2(8-6) + 2(4-3) - 2(4-3) + 0 + 0 - 0 + 0 - 0 = 1$$

$$C_{11} = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad C_{12} = -\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{13} = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0 \quad C_{14} = 0$$

$$C_{21} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{23} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} \quad C_{24} = 0$$

$$C_{31} = 0 \quad C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{vmatrix} \quad C_{33} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{vmatrix} \quad C_{34} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$C_{41} = 0 \quad C_{42} = 0 \quad C_{43} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \quad C_{44} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

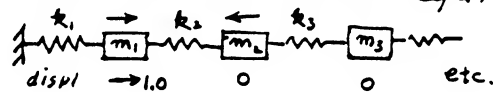
$$C'_{11} = C_{11} = 2, \quad C'_{12} = -1 \quad C'_{21} = -1, \quad C'_{22} = 2 \quad C'_{23} = -1$$

$$C'_{32} = -1 \quad C'_{33} = 2 \quad C'_{34} = -1 \quad C'_{43} = -1 \quad C'_{44} = 1$$

$$\therefore [k] = \frac{12EI}{l^3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

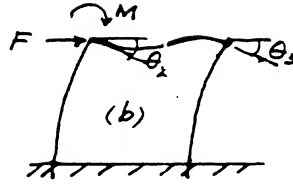
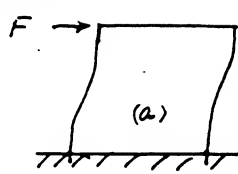
6-9

Give each mass a unit displ. keeping all others at zero. Then measure force required. ie for $x_1 = 1.0$



$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & 0 & 0 & \dots \\ -k_2 & (k_2 + k_3) & -k_3 & 0 & 0 & \dots \\ 0 & -k_3 & (k_3 + k_4) & -k_4 & 0 & \dots \\ 0 & 0 & -k_4 & (k_4 + k_5) & -k_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

6-10



$$(a) \quad k = 2 \times \frac{12EI}{l^3} = 24 \frac{EI}{l^3}$$

$$(b) \quad F = \frac{EI}{l^3} [24 - 6\theta_2 l - 6\theta_3 l] \quad \theta_2 = \theta_3 = \theta$$

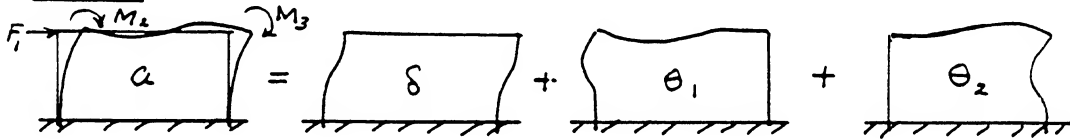
$$M = \frac{EI}{l^2} [-6 + 8\theta_2 l + 2\theta_3 l] = 0 \quad (\text{see Prob 6-11})$$

$$-6 + 10\theta l = 0 \quad \therefore \theta = \frac{6}{10} l$$

$$F = \frac{EI}{l^3} [24 - 12 \times \frac{6}{10}] = 16.8 \frac{EI}{l^3}$$

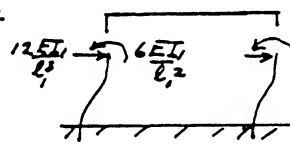
$$\text{Ratio } \frac{(a)}{(b)} = \frac{24}{16.8}$$

6-11

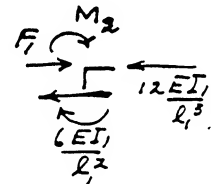


a is superposition of $\delta + \theta_1 + \theta_2$

δ mode

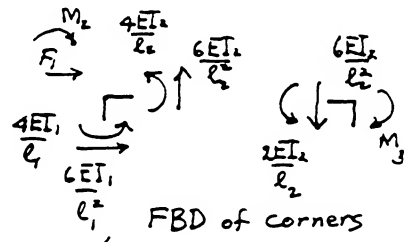
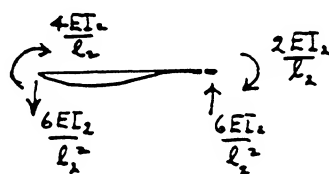
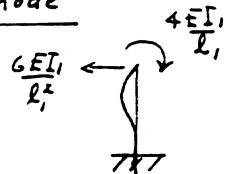


$$\therefore F_1 = 24 \frac{EI_1}{l_1^3} \delta$$



$$M_2 = M_3 = -6 \frac{EI_1}{l_1^2} \delta$$

θ_1 mode



FBD of corners

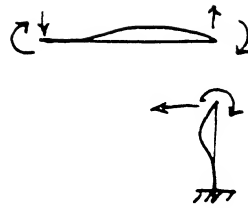
$$F_1 = -\frac{6EI_1}{l_1^2} \theta_1$$

$$M_2 = \left(\frac{4EI_1}{l_1} + \frac{4EI_1}{l_2} \right) \theta_1$$

$$M_3 = \left(\frac{2EI_1}{l_2} \right) \theta_1$$

6-11 Cont

θ_2 mode

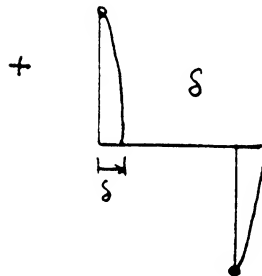
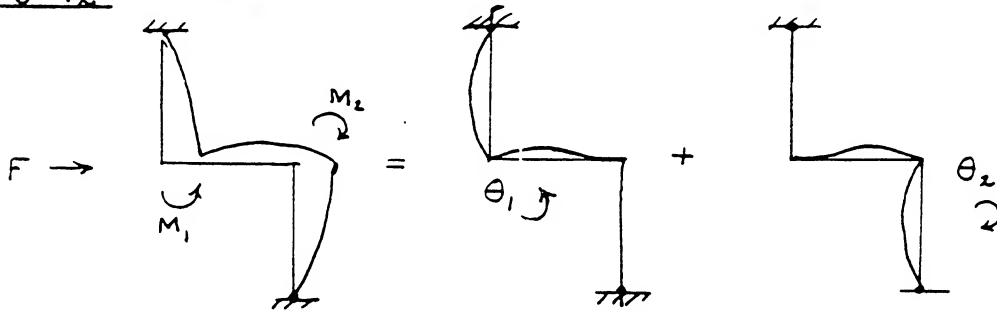


Same as mode θ_1 ,
but shifted to right corner

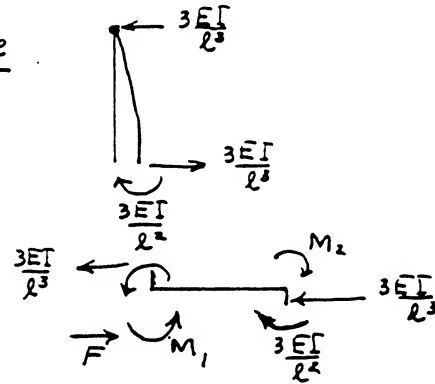
Sum results for δ , θ_1 & θ_2

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{24EI_1}{l_1^3} & -6\frac{EI_1}{l_1^2} & -6\frac{EI_1}{l_1^2} \\ -6\frac{EI_1}{l_1^2} & (4\frac{EI_1}{l_1} + 4\frac{EI_2}{l_2}) & 2\frac{EI_2}{l_2} \\ -6\frac{EI_1}{l_1^2} & 2\frac{EI_2}{l_2} & (4\frac{EI_1}{l_1} + 4\frac{EI_2}{l_2}) \end{bmatrix} \begin{Bmatrix} \delta \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

6-12



δ -mode



$$F = 6\frac{EI}{l^3} \delta$$

$$M_1 = -3\frac{EI}{l^2} \delta$$

$$M_2 = -3\frac{EI}{l^2} \delta$$

from corners

6-12 Cont

θ_1 -mode

$$F = -\frac{3EI}{l^2} \theta_1$$

$$M_1 = \left(\frac{3EI}{l} + \frac{4EI}{l} \right) (\theta_1)$$

$$M_2 = -\frac{2EI}{l} (\theta_1)$$

θ_2 -mode

this mode is same as above moved to right corner with θ_1 replaced by $+\theta_2$

Add results

$$\begin{Bmatrix} F \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{6EI}{l^3} & -\frac{3EI}{l^2} & -\frac{3EI}{l^2} \\ -\frac{3EI}{l^2} & \left(\frac{3EI}{l} + \frac{4EI}{l} \right) & -\frac{2EI}{l} \\ -\frac{3EI}{l^2} & -\frac{2EI}{l} & \left(\frac{3EI}{l} + \frac{4EI}{l} \right) \end{bmatrix} \begin{Bmatrix} \delta \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

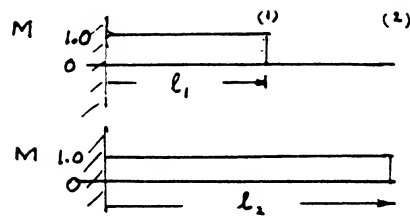
Since no moments were applied at corners $M_1 = M_2 = 0$

$$\begin{cases} 0 = -\frac{3EI}{l^2} \delta + \frac{7EI}{l} \theta_1 - \frac{2EI}{l} \theta_2 \\ 0 = -\frac{3EI}{l^2} \delta - \frac{2EI}{l} \theta_1 + \frac{7EI}{l} \theta_2 \end{cases} \left. \begin{array}{l} \text{Subtr. we find} \\ \theta_1 = \theta_2 = \theta \\ \text{as expected by symmetry} \end{array} \right\}$$

$$M_1 = -\frac{3EI}{l^2} \delta + \frac{5EI}{l} \theta = 0 \quad \therefore \theta = \frac{3\delta}{5l}$$

$$F = \frac{6EI}{l^3} \delta - \frac{6EI}{l^2} \left(\frac{3\delta}{5l} \right) = \underline{\underline{2.40 \frac{EI}{l^3} \delta}}$$

6-13 Moment Diagram



$$EI \theta_{21} = EI \theta_{12} = M_1 l_1 = M_2 l_2$$

$$M_1 = M_2 = 1$$

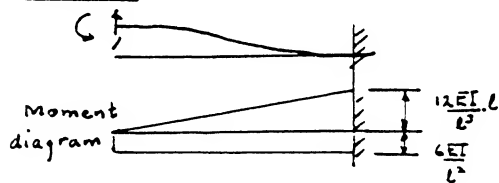
$$\therefore \theta_{12} = \theta_{21}$$

By Work done

$$M_1 \text{ 1st followed by } M_2 \quad W = \frac{1}{2} M_1 \theta_{11} + \frac{1}{2} M_2 \theta_{22} + M_1 \theta_{12}$$

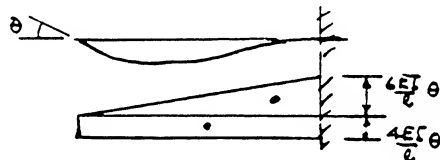
$$M_2 \text{ 1st " " } M_1 \quad W = \frac{1}{2} M_2 \theta_{22} + \frac{1}{2} M_1 \theta_{11} + M_2 \theta_{21}$$

6-14



Difference in slope = 0

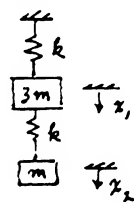
$$\left(\frac{1}{2} \cdot \frac{12EI}{l^2} \cdot l \right) - \left(\frac{6EI}{l^2} \cdot l \right) = 0$$



Difference in defl. = 0

$$\left(\frac{1}{2} \cdot \frac{6EI}{l^2} \right) \frac{2}{3} l - \left(\frac{4EI}{l^2} \cdot l \right) \frac{l}{2} = 0$$

6-15



$$m \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{Let } \lambda = \frac{m\omega^2}{k}$$

$$\begin{vmatrix} (2-3\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \quad \lambda^2 - \frac{5}{3}\lambda + \frac{1}{3} = 0$$

$$\text{Adj. matrix} = \begin{bmatrix} (1-\lambda) & 1 \\ 1 & (2-3\lambda) \end{bmatrix}$$

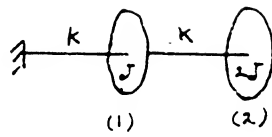
$$\lambda = 0.8333 \pm \sqrt{.6944 - .3333}$$

$$= 0.8333 \pm 0.6010 = \begin{cases} 0.2323 \\ 1.4343 \end{cases}$$

$$\text{Subst. } \lambda_1 \text{ into either column for 1st mode} = \begin{Bmatrix} .7677 \\ 1.000 \end{Bmatrix}$$

$$\text{Subst } \lambda_2 \text{ into " " " 2nd mode} = \begin{Bmatrix} -.4343 \\ 1.000 \end{Bmatrix}$$

6-16



$$\begin{bmatrix} J & 0 \\ 0 & 2J \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \frac{\omega^2 J}{K}$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (1-2\lambda) \end{vmatrix} = 0 \quad 2\lambda^2 - 5\lambda + 1 = 0$$

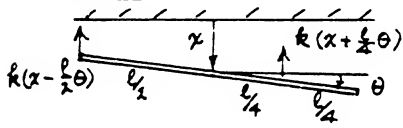
$$\lambda = \frac{5}{4} \pm \sqrt{\frac{25}{16} - \frac{8}{16}} = \begin{cases} 0.2192 \\ 2.2808 \end{cases}$$

Adj. matrix = $\begin{bmatrix} (1-2\lambda) & 1 \\ 1 & (2-\lambda) \end{bmatrix}$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.000 \\ 1.781 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -2.2808 \end{Bmatrix}$$

6-17



$$m \frac{l^2}{12} \ddot{\theta} = \frac{1}{2} k(x - \frac{l}{2}\theta) - \frac{l}{4} k(x + \frac{l}{2}\theta) = \frac{kl}{4} x - \frac{5kl}{16} \theta$$

$$m \ddot{x} = -k(x - \frac{l}{2}\theta) - k(x + \frac{l}{2}\theta) = -2kx + \frac{kl}{4} \theta$$

$$m \begin{bmatrix} \frac{l^2}{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + k \begin{bmatrix} \frac{5}{16} l^2 & -\frac{l}{4} \\ -\frac{l}{4} & 2 \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \frac{\omega^2 m}{k}$

$$\begin{vmatrix} (\frac{5}{16} l^2 - \frac{l^2}{12} \lambda) - \frac{l}{4} & -\frac{l}{4} \\ -\frac{l}{4} & (2 - \lambda) \end{vmatrix} = 0$$

$$\lambda^2 - \frac{23}{4} \lambda + \frac{9 \times 12}{16} = 0$$

$$\lambda = \frac{23}{8} \pm \sqrt{\frac{97}{16}}$$

$$\lambda = \frac{1}{8} (23 \pm 9.849) = \begin{cases} 1.644 \\ 4.106 \end{cases}$$

$$\frac{l\theta}{x} = 4(2 - \lambda) = \begin{cases} 1.424 \\ -8.424 \end{cases}$$

$$\begin{Bmatrix} \theta \\ x \end{Bmatrix}_1 = \begin{Bmatrix} 1.424/l \\ 1.000 \end{Bmatrix}$$

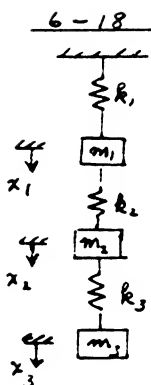
$$\begin{Bmatrix} \theta \\ x \end{Bmatrix}_2 = \begin{Bmatrix} -8.424/l \\ 1.000 \end{Bmatrix}$$

$$P = \begin{bmatrix} 1.424/l & -8.424/l \\ 1.000 & 1.000 \end{bmatrix}$$

To show that $P' K P = \text{diagonal}$

$$\begin{bmatrix} 1.424/l & 1.0 \\ -8.424/l & 1.0 \end{bmatrix} \begin{bmatrix} \frac{5}{16} l^2 & -\frac{l}{4} \\ -\frac{l}{4} & 2 \end{bmatrix} \begin{bmatrix} 1.424/l & -8.424/l \\ 1.000 & 1.000 \end{bmatrix} = \begin{bmatrix} 1.424/l & 1.0 \\ -8.424/l & 1.0 \end{bmatrix} \begin{bmatrix} 1.958 & -2.8825 \\ 1.644 & 4.106 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9217 & 0.0013 \\ 0.0013 & 28.388 \end{bmatrix} \approx \begin{bmatrix} 1.9217 & 0 \\ 0 & 28.388 \end{bmatrix}$$



Place unit load at 1.

$$a_{11} = \frac{1}{k_1} = a_{21} = a_{31} = a_{12} = a_{13}$$

Place unit load at 2

$$a_{22} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right) = a_{32} = a_{23}$$

Place unit load at 3

$$a_{33} = \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)$$

Eq. of motion:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega^2 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

6-19

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{Let } \lambda = \frac{m\omega^2}{k}$$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (2-2\lambda) \end{vmatrix} = 0 \quad \lambda^2 - 3\lambda + \frac{3}{2} = 0$$

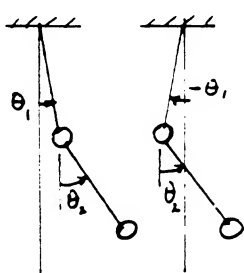
$$\lambda = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{3}{2}} = 1.5 \pm 0.866 = \begin{cases} 0.634 \\ 2.366 \end{cases}$$

$$\frac{x_1}{x_2} = \frac{2(1-\lambda)}{1} = \begin{cases} 0.732 \\ -2.732 \end{cases} \quad \therefore P = \begin{bmatrix} 0.732 & -2.732 \\ 1.000 & 1.000 \end{bmatrix}$$

$$P' M P \ddot{y} + P' K P y = 0$$

$$\begin{bmatrix} 2.535 & 0 \\ 0 & 9.48 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 1.606 & 0 \\ 0 & 22.33 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{uncoupled.}$$

6-20



$$\frac{l}{g} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

\therefore dynamically coupled.

From Prob. 5-10

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.707 \\ 1.00 \end{Bmatrix} \quad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} -0.707 \\ 1.00 \end{Bmatrix}$$

6-20 Cont.

Gen. mass

$$\text{mode 1} \quad (.707 \quad 1.0) \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} .707 \\ 1.0 \end{Bmatrix} = 3.414$$

$$\text{mode 2} \quad (-.707 \quad 1.0) \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} = 0.586$$

$$\tilde{P} = \begin{bmatrix} .707 & -.707 \\ 1.0 & 1.0 \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} .207 & -1.208 \\ .293 & 1.707 \end{bmatrix}$$

$$\tilde{P}' M \tilde{P} = \begin{bmatrix} .207 & .293 \\ -1.207 & 1.707 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .207 & -1.207 \\ .293 & 1.707 \end{bmatrix} = \begin{bmatrix} 0.293 & 0 \\ 0 & 1.707 \end{bmatrix}$$

$$\tilde{P}' K \tilde{P} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} .1714 & 0 \\ 0 & 5.82 \end{bmatrix}$$

Normalized eq. of motion becomes

$$\begin{bmatrix} .293 & 0 \\ 0 & 1.707 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} .1714 & 0 \\ 0 & 5.82 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \therefore \text{uncoupled}$$

6-21

See Prob. 6-11

$$\begin{bmatrix} (m_1 + m_2) & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{24EI_1}{l_1^3} & -\frac{6EI_1}{l_1^2} & -\frac{6EI_1}{l_1^2} \\ -\frac{6EI_1}{l_1^2} & (\frac{4EI_1}{l_1} + \frac{4EI_2}{l_2}) & \frac{2EI_2}{l_2} \\ -\frac{6EI_1}{l_1^2} & \frac{2EI_2}{l_2} & (\frac{4EI_1}{l_1} + \frac{4EI_2}{l_2}) \end{bmatrix} \begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assign numbers before solving.

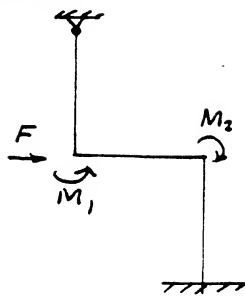
6-22

See Prob 6-12

$$\begin{bmatrix} (m_1 + m_2) & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \frac{6EI}{l^3} & -\frac{3EI}{l^2} & -\frac{3EI}{l^2} \\ -\frac{3EI}{l^2} & \frac{7EI}{l} & -\frac{2EI}{l} \\ -\frac{3EI}{l^2} & -\frac{2EI}{l} & \frac{7EI}{l} \end{bmatrix} \begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

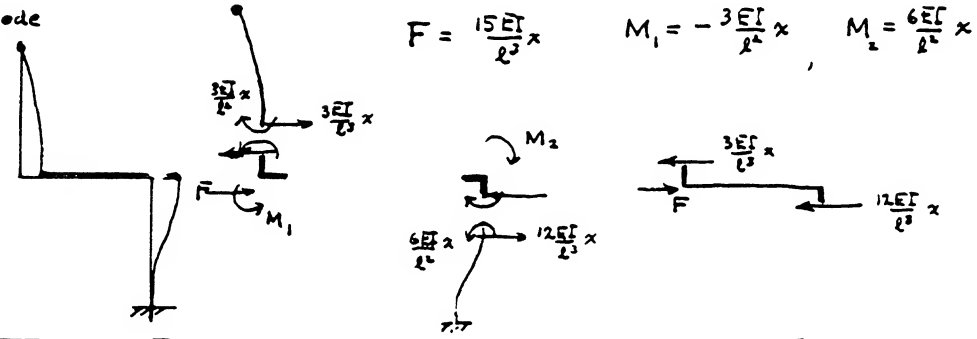
Assign numbers before solving

6-23

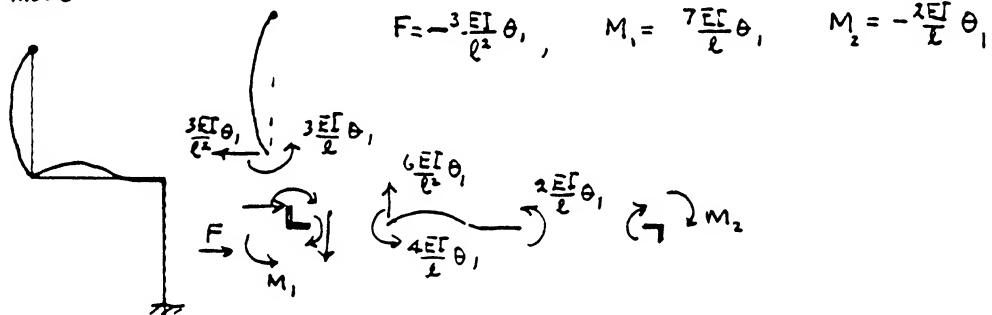


The problem is the superposition of the three modes below. Use Figure 6.4-2 and examine the free-body diagrams of members and corners.

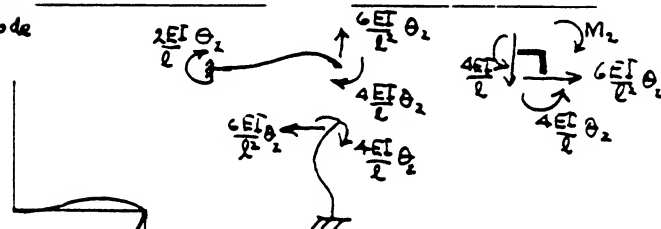
x -mode



θ_1 -mode



θ_2 -mode



$$\begin{Bmatrix} F \\ M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{15EI}{l^3} & -\frac{3EI}{l^2} & \frac{6EI}{l^2} \\ -\frac{3EI}{l^2} & \frac{7EI}{l} & -\frac{2EI}{l} \\ \frac{6EI}{l^2} & -\frac{2EI}{l} & \frac{8EI}{l} \end{bmatrix} \begin{Bmatrix} x \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

6-24

$$m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + c \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} \sin \omega t$$

\therefore Damping is not proportional

6-25
char. eq. $\lambda = \frac{\omega^2 m}{k}$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -1 & (2-\lambda) \end{vmatrix} = 0 \quad \lambda = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \tilde{P} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Let $x = \tilde{P} y$ in eq. Prob. 6-24 and premult. by \tilde{P}'

$$\tilde{P}' M \tilde{P} \ddot{y} + \tilde{P}' C \tilde{P} \dot{y} + \tilde{P}' k \tilde{P} y = \tilde{P}' F$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \frac{c}{2m} \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \frac{k}{m} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ -F_0 \end{Bmatrix} \frac{\sin \omega t}{\sqrt{2m}}$$

\therefore coupled only by damping

$$\ddot{y}_1 + \frac{c}{2m} (\dot{y}_1 - \dot{y}_2) + \frac{k}{m} y_1 = \frac{F_0}{\sqrt{2m}} \sin \omega t$$

$$\ddot{y}_2 + \frac{c}{2m} (-\dot{y}_1 + 5\dot{y}_2) + \frac{3k}{m} y_2 = \frac{-F_0}{\sqrt{2m}} \sin \omega t$$

Steady State solution

$$\left(\frac{k}{m} - \omega^2 + i \frac{c}{2m} \omega \right) Y_1 - i \left(\frac{c}{2m} \omega \right) Y_2 = \frac{F_0}{\sqrt{2m}}$$

$$-i \left(\frac{c}{2m} \omega \right) Y_1 + \left(\frac{3k}{m} - \omega^2 + i \frac{5c}{2m} \omega \right) Y_2 = \frac{-F_0}{\sqrt{2m}}$$

$$Y_1 = \frac{\begin{vmatrix} 1 & i\omega \frac{c}{2m} \\ -1 & \left(\frac{3k}{m} - \omega^2 + i\omega \frac{5c}{2m} \right) \end{vmatrix} \frac{F_0}{\sqrt{2m}}}{\left(\frac{k}{m} - \omega^2 + i \frac{c}{2m} \omega \right) \left(\frac{3k}{m} - \omega^2 + i \omega \frac{5c}{2m} \right) + \left(\frac{\omega c}{2m} \right)^2}$$

6-26

$$\text{Let } \omega_0^2 = \frac{k}{m}, \quad \alpha = \frac{k_1}{C}, \quad \beta = \frac{k_1}{m}$$

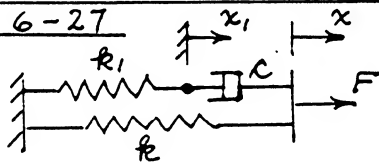
The equations are rewritten as

$$\ddot{x} = -\omega_0^2 x - \beta x_1 + \frac{F}{m}$$

$$\dot{x}_1 = \dot{x} - \alpha x_1$$

$$\left. \begin{array}{l} \text{Let } x_1 = z_1 \\ \dot{x}_1 = \dot{z}_1 \\ x = z_2 \\ \dot{x} = \dot{z}_2 = \dot{z}_3 \end{array} \right\} \text{ then } \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{Bmatrix} = \begin{bmatrix} -\alpha & 0 & 1 \\ 0 & 0 & 1 \\ -\beta & -\omega_0^2 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ \frac{F}{m} \end{Bmatrix}$$

6-27



$$F = kx + C(\dot{x} - \dot{x}_1) \quad (a)$$

$$k_1 x_1 = C(\dot{x} - \dot{x}_1) \quad (b)$$

Assume F to be harmonic; then from (b)

$$x_1 = \frac{i\omega C}{k_1 + i\omega C} x = \frac{i(\omega C/k_1)}{1 + i(\omega C/k_1)} x$$

Subst. into (a)

$$F = kx + i\omega C \left[1 - \frac{i(\omega C/k_1)}{1 + i(\omega C/k_1)} \right] x$$

$$= \frac{[k(1 + \frac{i\omega C}{k_1}) + i\omega C]}{(1 + \frac{i\omega C}{k_1})} \cdot \frac{(1 - \frac{i\omega C}{k_1})}{(1 - \frac{i\omega C}{k_1})} x$$

$$= \left\{ \frac{k + (k+k_1)(\frac{\omega C}{k_1})^2}{1 + (\frac{\omega C}{k_1})^2} + \frac{i\omega C}{1 + (\frac{\omega C}{k_1})^2} \right\} x$$

$$= \left\{ k_{eq} + i\omega C_{eq} \right\} x$$

$$= \frac{k + (k+k_1)(\frac{\omega C}{k_1})^2}{1 + (\frac{\omega C}{k_1})^2} \left\{ 1 + i \frac{\omega C}{k + (k+k_1)(\frac{\omega C}{k_1})^2} \right\} x = k^* (1 + i\gamma) x$$

6-28

Refer to Prob. 6-16

$$X_1 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1.00 \\ 1.781 \end{Bmatrix} \quad X_2 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1.00 \\ -0.2808 \end{Bmatrix}$$

$$\begin{aligned} X_1' K X_2 &= (1.00 \quad 1.781) K \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.00 \\ -0.2808 \end{Bmatrix} \\ &= (1.00 \quad 1.781) \begin{Bmatrix} 2.2808 \\ -1.2808 \end{Bmatrix} = 2.2808 - 2.2811 \\ &= -0.0003 \approx 0 \end{aligned}$$

6-29

$$K \phi_s = \omega_s^2 M \phi_s \quad \begin{array}{l} \phi = \text{normal mode} \\ \times \text{ by } K M^{-1} \end{array}$$

$$K M^{-1} K \phi_s = \omega_s^2 K M^{-1} M \phi_s = \omega_s^2 K \phi_s$$

$$\underline{\underline{\phi_n' [K M^{-1} K] \phi_s = \omega_s^2 [\phi_n' K \phi_s] = 0 \quad \text{for } n \neq s}}$$

$$\text{Repeat } K M^{-1} K \phi_s = \omega_s^2 K \phi_s \quad \times \text{ by } K M^{-1}$$

$$K M^{-1} K M^{-1} K \phi_s = \omega_s^2 [K M^{-1} K] \phi_s \quad \times \text{ by } \phi_n'$$

$$\underline{\underline{\phi_n' [K M^{-1}]^2 K \phi_s = \omega_s^2 \phi_n' [K M^{-1} K] \phi_s = 0 \quad n \neq s}}$$

$$\text{Repeat to obtain } \phi_n' [K M^{-1}]^h K \phi_s = 0$$

6-30

$$K \phi_s = \omega_s^2 M \phi_s, \quad M K^{-1} K \phi_s = \omega_s^2 M K^{-1} M \phi_s$$

$$\therefore M \phi_s = \omega_s^2 M K^{-1} M \phi_s \quad \text{ie } K^{-1} K = I$$

$$\phi_n' M \phi_s = \omega_s^2 \phi_n' [M K^{-1} M] \phi_s = 0 \quad \text{ie } \phi_n' M \phi_s = 0 \quad n \neq s$$

Repeat

$$M K^{-1} M \phi_s = \omega_s^2 [M K^{-1}]^2 M \phi_s$$

$$\underline{\underline{\phi_n' M K^{-1} M \phi_s = \omega_s^2 \phi_n' [M K^{-1}]^2 M \phi_s = 0 \quad n \neq s}}$$

Repeat h times.

6-31 Refer to Ex. 6.10-1 for ϕ_i & ω_i

For second mode

$$m_{22} \ddot{q}_2 + c_{22} \dot{q}_2 + k_{22} q_2 = -\ddot{u}_0(t) \sum_{i=1}^{10} m_i \phi_2(x_i)$$

$$m_{22} = \sum_{i=1}^{10} m_i \phi_2^2(x_i) = 5.5235 \text{ m}$$

$$c_{22} = 2\zeta_2 \omega_2 m_{22} = 2\zeta_2 (0.1451 \sqrt{\frac{k}{m}}) m_{22} = 0.8902 \zeta_2 \sqrt{\frac{k}{m}} m_{22}$$

$$k_{22} = \omega_2^2 m_{22} = (0.1981 \frac{k}{m}) m_{22}$$

$$\sum_{i=1}^{10} m_i \phi_2(x_i) = -2.2470 \text{ m}$$

$$\begin{aligned} \therefore \ddot{q}_2 + 0.8902 \zeta_2 \sqrt{\frac{k}{m}} \dot{q}_2 + 0.1981 \frac{k}{m} q_2 &= \frac{2.247}{5.5235} \ddot{u}_0(t) \\ &= 0.4068 \ddot{u}_0(t) \end{aligned}$$

For 3rd mode

$$m_{33} = 8.5957 \text{ m}, \quad c_{33}/m_{33} = 2\zeta_3 (0.7307 \sqrt{\frac{k}{m}}) = 1.4614 \zeta_3 \sqrt{\frac{k}{m}}$$

$$k_{33} = 0.5339 \frac{k}{m}, \quad \sum_{i=1}^{10} m_i \phi_3(x_i) = 2.8095 \text{ m}$$

$$\ddot{q}_3 + 1.4614 \zeta_3 \sqrt{\frac{k}{m}} \dot{q}_3 + 0.5339 \frac{k}{m} q_3 = -0.3268 \ddot{u}_0(t)$$

6-32 From Ex. 6.10-1

$$\ddot{q}_1 + 0.299 \zeta_1 \sqrt{\frac{k}{m}} \dot{q}_1 + 0.02235 \frac{k}{m} q_1 = -1.2672 \ddot{u}_0(t)$$

$$\omega_1^2 = 0.02235 \frac{k}{m}, \quad \omega_1 = 0.1495 \sqrt{\frac{k}{m}} = \frac{2\pi}{T_1}$$

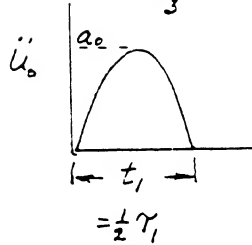
$$\therefore T_1 = \frac{2\pi}{0.1495 \sqrt{\frac{k}{m}}} = 42.028 \sqrt{\frac{m}{k}}$$

6-32 Cont.

From Ex. 6.10-1

$$\omega_2 = \frac{2\pi}{\gamma_2} = 0.4451 \sqrt{\frac{k}{m}}, \quad \gamma_2 = 14.1168 \sqrt{\frac{m}{k}}$$

$$\omega_3 = 0.7307 \sqrt{\frac{k}{m}}, \quad \gamma_3 = 8.5989 \sqrt{\frac{m}{k}}$$



See Fig 4.5-5 for shock spectrum

$$\frac{t_1}{\gamma_1} = 0.50 \quad \left(\frac{xk}{F_0} \right)_{\max} = 1.5$$

$$\frac{t_1}{\gamma_1} \frac{\gamma_1}{\gamma_2} = \frac{t_1}{\gamma_2} = .5 \frac{42.028}{14.116} = 1.4886 \therefore \left(\frac{xk}{F_0} \right)_{\max} = 1.5$$

$$\frac{t_1}{\gamma_3} = .50 \frac{42.028}{8.5989} = 2.4438 \therefore \left(\frac{xk}{F_0} \right)_{\max} = 1.13$$

The right side of the DEs are:

$$-\ddot{u}_0(t) \frac{\sum m \phi_i}{\sum m \phi_i^2} = -1.2672 \ddot{u}_0 = \frac{F_0}{m} \quad \text{by comparison}$$

with $\ddot{q} + 2\zeta \omega_n \dot{q} + \omega_n^2 q = \frac{F_0}{m}$

$$\therefore F_0 = -1.2672 m a_0 \quad \therefore \text{in place of } \left(\frac{xk}{F_0} \right)_{\max}$$

$$\text{we use } \left(\frac{qk}{F_0} \right)_{\max} = \frac{qk}{-1.2672 m a_0} \quad \text{for mode 1.} = 1.5$$

$$\therefore (q_1)_{\max} = -1.5 \times 1.2672 \frac{m}{k} a_0 = \underline{\underline{-1.9008 \frac{m a_0}{k}}}$$

Similarly for 2nd mode & 3rd mode

$$(q_2)_{\max} = 1.5 \times .4068 \frac{m a_0}{k} = \underline{\underline{0.6102 \frac{m a_0}{k}}}$$

$$(q_3)_{\max} = 1.13 \times (-.3263) \frac{m a_0}{k} = \underline{\underline{-0.3693 \frac{m a_0}{k}}}$$

6-32 Cont:

$$x(t) = \phi_1 q_1 + \phi_2 q_2 + \phi_3 q_3 + \dots$$

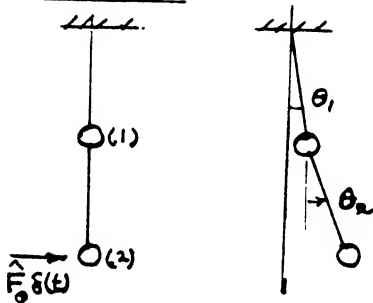
but at 10th floor $\phi_i = 1.0$

$$\therefore x(t) = q_1 + q_2 + q_3$$

From Eq. (6.10-7)

$$\begin{aligned} |x(10)|_{\max} &= (q_1)_{\max} + \sqrt{(q_2)_{\max}^2 + (q_3)_{\max}^2} \\ &= 1.90 + \sqrt{.610^2 + .369^2} \\ &= 1.90 + 0.711 = 2.61 \frac{m\Omega_0}{k} \end{aligned}$$

6-33



Impulse = change in momentum

At $t=0$ mass at (2) will acquire a velocity of $\frac{\hat{F}_0}{m} = v(0) = l \dot{\theta}_2(0)$

$$\therefore \dot{\theta}_2(0) = \frac{\hat{F}}{ml} \quad \dot{\theta}_1(0) = 0$$

$$\theta = \phi_1 q_1 + \phi_2 q_2$$

$$\therefore \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} q_1 + \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} q_2$$

$$= \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A \sin .764 \sqrt{\frac{g}{l}} t + \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B \sin 1.85 \sqrt{\frac{g}{l}} t$$

$$\sqrt{\frac{l}{g}} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = .764 \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A \cos .764 \sqrt{\frac{g}{l}} t + 1.85 \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B \cos 1.85 \sqrt{\frac{g}{l}} t$$

At $t=0$

$$\sqrt{\frac{l}{g}} \begin{Bmatrix} 0 \\ \frac{\hat{F}}{ml} \end{Bmatrix} = .764 \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} A + 1.85 \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} B$$

6-33 Cont

$$\therefore 0 = (.764)(.707)A - (1.85)(.707)B$$

$$B = 0.413 A$$

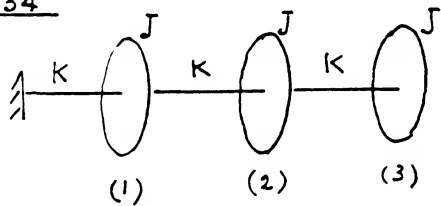
$$\sqrt{\frac{l}{g}} \frac{\hat{F}_0}{ml} = .764 A + 1.85 (.413 A) = 1.528 A$$

$$\therefore A = 0.6544 \sqrt{\frac{l}{g}} \frac{\hat{F}}{ml}$$

$$B = 0.2703 \sqrt{\frac{l}{g}} \frac{\hat{F}}{ml}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \sqrt{\frac{l}{g}} \frac{\hat{F}}{ml} \left[(0.6544) \begin{Bmatrix} .707 \\ 1.00 \end{Bmatrix} \sin .764 \sqrt{\frac{g}{l}} t + (.2703) \begin{Bmatrix} -.707 \\ 1.00 \end{Bmatrix} \sin 1.85 \sqrt{\frac{g}{l}} t \right]$$

6-34



$$\lambda = \frac{J \omega^2}{K}$$

$$\phi = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$J \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + K \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_0 u(t) \end{Bmatrix}$$

$$\text{Let } \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = P \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \text{ \& premultiply by } P'$$

$$P' J P \ddot{q} + P' K P q = P' \{M\}$$

$$m_{ii} \ddot{q}_i + k_{ii} q_i = M_0 u(t) \theta_3 i$$

$$\ddot{q}_i + \omega_i^2 q_i = \frac{M_0 (\theta_3)_i u(t)}{m_{ii}}$$

$$m_{11} = \phi_1' J \phi_1 = J (.328^2 + .591^2 + .737^2) = 1.00 J$$

$$\omega_1^2 = .198 \frac{K}{J}$$

$$m_{22} = J (.737^2 + .328^2 + .591^2) = 1.00 J$$

$$\omega_2^2 = 1.555 \frac{K}{J}$$

$$m_{33} = 1.00 J$$

$$\omega_3^2 = 3.247 \frac{K}{J}$$

$$\text{Right side of eq } \frac{M_0 (\theta_3)_i u(t)}{m_{ii}} =$$

$$\frac{.737}{J} M_0 u(t)$$

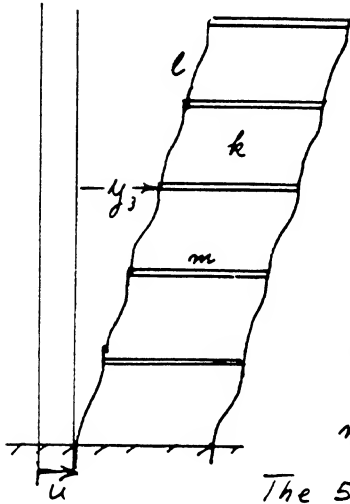
$$- \frac{.591}{J} M_0 u(t)$$

$$\text{and } \frac{.328}{J} M_0 u(t)$$

6-35 Cont.

$$\frac{\sum m \phi_1}{\sum m \phi_1^2} = 2.097$$

$$\frac{\sum m \phi_2}{\sum m \phi_2^2} = 0.6602$$



For $K_R = \infty$, $\theta = 0$ and we have only translation of ground plus elastic translation of each floor. The more general case of $\theta \neq 0$ should be deferred until Ch 8, with Lagrange's Eq.

Write force eq. for one mass (say 3rd floor)

$$m(\ddot{u} + \ddot{y}_3) = -k(y_3 - y_2) + k(y_4 - y_3)$$

The 5 equations like the above can be written in matrix form

$$m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = -m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ddot{u}$$

or $M \ddot{y} + K y = -M \ddot{u}$

Let $y = \phi_1 q_1 + \phi_2 q_2$ & decouple by $P' K P$

or $\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = P q$

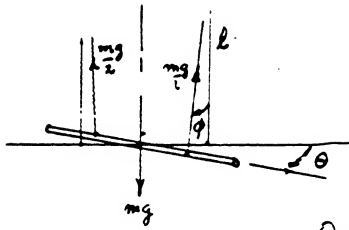
$$P' M P \ddot{q} + P' K P q = -P' M \ddot{u}$$

The modal eqs. become

$$\ddot{q}_1 + \omega_1^2 q_1 = -\frac{\sum m \phi_1}{\sum m \phi_1^2} \ddot{u}$$

$$\ddot{q}_2 + \omega_2^2 q_2 = -\frac{\sum m \phi_2}{\sum m \phi_2^2} \ddot{u}$$

6-36



$$\phi \approx \frac{a\theta}{2L}$$

Torsional Oscil.

$$J\ddot{\theta} = -2mg \frac{a\theta}{2L} \frac{a}{2} = -mg \frac{a^2}{4L} \theta$$

$$\therefore \omega_r^2 = \frac{mg a^2}{4LJ} = \frac{mg a^2}{4L \frac{m L^2}{12}} = 3 \frac{g}{L} \left(\frac{a}{L}\right)^2$$

Out-of plane oscillation is that of simple pendulum with $\omega^2 = \frac{g}{L}$. For $\omega^2 = \omega_r^2$

$$\frac{g}{L} = 3 \frac{g}{L} \left(\frac{a}{L}\right)^2 \quad \therefore \frac{a}{L} = \frac{1}{\sqrt{3}}$$

With small eccentricity to excite torsional oscil, there will be beating (see Prob. 1-4)

6-37

Modal damping given as $\zeta_1 = .05$, $\zeta_2 = .13$

From Eq. (6.9-9)

$$2\zeta_1 \omega_1 = \alpha + \beta \omega_1^2$$

$$\omega_1 = .445 \sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.247 \sqrt{\frac{k}{m}}$$

$$2\zeta_2 \omega_2 = \alpha + \beta \omega_2^2$$

$$\omega_3 = 1.802 \sqrt{\frac{k}{m}}$$

$$2(\zeta_1 \omega_1 - \zeta_2 \omega_2) = \beta(\omega_1^2 - \omega_2^2)$$

$$\therefore \beta = \frac{2(\zeta_2 \omega_2 - \zeta_1 \omega_1)}{\omega_2^2 - \omega_1^2}, \quad \alpha = \frac{2\omega_1 \omega_2 (\zeta_1 \omega_2 - \zeta_2 \omega_1)}{\omega_2^2 - \omega_1^2}$$

$$\therefore \beta = \frac{2(.05 \times .445 - .13 \times 1.247) \sqrt{\frac{k}{m}}}{(.198 - 1.555) \sqrt{\frac{k}{m}}} = \underline{\underline{0.2061 \sqrt{\frac{m}{k}}}}$$

$$\alpha = \frac{2 \times .445 \times 1.247 (.05 \times 1.247 - .13 \times .445) \sqrt{\frac{k}{m}}}{1.357} = \underline{\underline{0.00370 \sqrt{\frac{k}{m}}}}$$

For 3rd mode

$$2\zeta_3 \omega_3 = \alpha + \beta \omega_3^2$$

$$\zeta_3 = \frac{\alpha + \beta \omega_3^2}{2\omega_3}$$

$$\zeta_3 = \frac{.00370 + .2061 \times 3.247}{2 \times 1.802} = \underline{\underline{0.1867}}$$

6-38 $X_i' M X_j = (- \dots)_i m [I] \begin{Bmatrix} - \\ - \\ - \end{Bmatrix}_{j,i} = m (- \dots)_i \begin{Bmatrix} - \\ - \\ - \end{Bmatrix}_{j,i}$

With the modes given $X_i' M X_j = \begin{cases} 0 & \text{for } j \neq i \\ 1 & \text{for } j = i \end{cases}$

6-39 See Ex. 6.6-1 $C_i = \frac{\phi_i^T M u(0)}{\phi_i^T M \phi_i}$

From data given $\phi_i^T M \phi_i = 1.0 \text{ m}$

$$\therefore C_1 = \frac{(.737 \ .591 \ .328)}{1.0 \text{ m}} \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = .3914$$

$$C_2 = (-.591 \ .328 \ .737) \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = -.1890 = -.4829 C_1$$

$$C_3 = (.328 \ -.737 \ .591) \begin{Bmatrix} .520 \\ -.100 \\ .205 \end{Bmatrix} = .3654 = .9336 C_1$$

6-40

$$X(t) = \sum_i X_i (A_i \sin \omega_i t + B_i \cos \omega_i t)$$

$$\dot{X}(t) = \sum_i \omega_i X_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

$$\dot{X}(0) = \sum_i \omega_i X_i A_i$$

$$X_j' M \dot{X}(0) = \sum_i \omega_i X_j' M X_i A_i = \omega_j X_j' M X_j A_j$$

$$\therefore A_j = \frac{X_j' M \dot{X}(0)}{\omega_j X_j' M X_j}$$

$$B_j = \frac{X_j' M X(0)}{X_j' M X_j}$$

6-41 The bending of the shaft is measured from the slope of the bearing, which is β . Thus y and θ in Example 6.1-4 must first be replaced by η and $(\theta - \beta)$ respectively.

$$\begin{Bmatrix} \eta \\ \theta - \beta \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix}$$

From geometry Fig P6-41

$$y = \frac{P}{k} + \beta l + \eta$$

$$M_B = K\beta = Pl + M \quad \therefore \beta = \frac{Pl}{K} + \frac{M}{K}$$

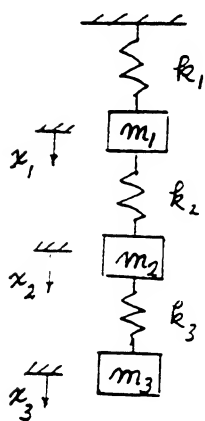
$$\text{and } \eta = y - \frac{P}{k} - \beta l = y - \frac{P}{k} - \frac{Pl^2}{K} - \frac{Ml}{K}$$

$$\therefore y - \frac{P}{k} - \frac{Pl^2}{K} - \frac{Ml}{K} = a_{11}P + a_{12}M$$

$$\theta - \frac{Pl}{K} - \frac{M}{K} = a_{21}P + a_{22}M$$

$$\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{bmatrix} (a_{11} + \frac{1}{k} + \frac{l^2}{K}) & (a_{12} + \frac{l}{K}) \\ (a_{21} + \frac{l}{K}) & (a_{22} + \frac{1}{K}) \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} \begin{Bmatrix} P \\ M \end{Bmatrix}$$

6-42



$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$M\ddot{X} + KX = 0$$

$$[-\omega^2 I + M^{-1}K]X = 0$$

$$[A - \lambda I]X = 0$$

$$\text{where } A = M^{-1}K$$

6-42 Cont.

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m_1}(k_1+k_2) & -\frac{1}{m_1}k_2 & 0 \\ -\frac{1}{m_2}k_2 & \frac{1}{m_2}(k_2+k_3) & -\frac{1}{m_2}k_3 \\ 0 & -\frac{1}{m_3}k_3 & \frac{1}{m_3}k_3 \end{bmatrix} \therefore \text{not symmetric}$$

For symmetric matrix proceed as follows

$$Q = M^{\frac{1}{2}} = \begin{bmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix} = Q^{-T}$$

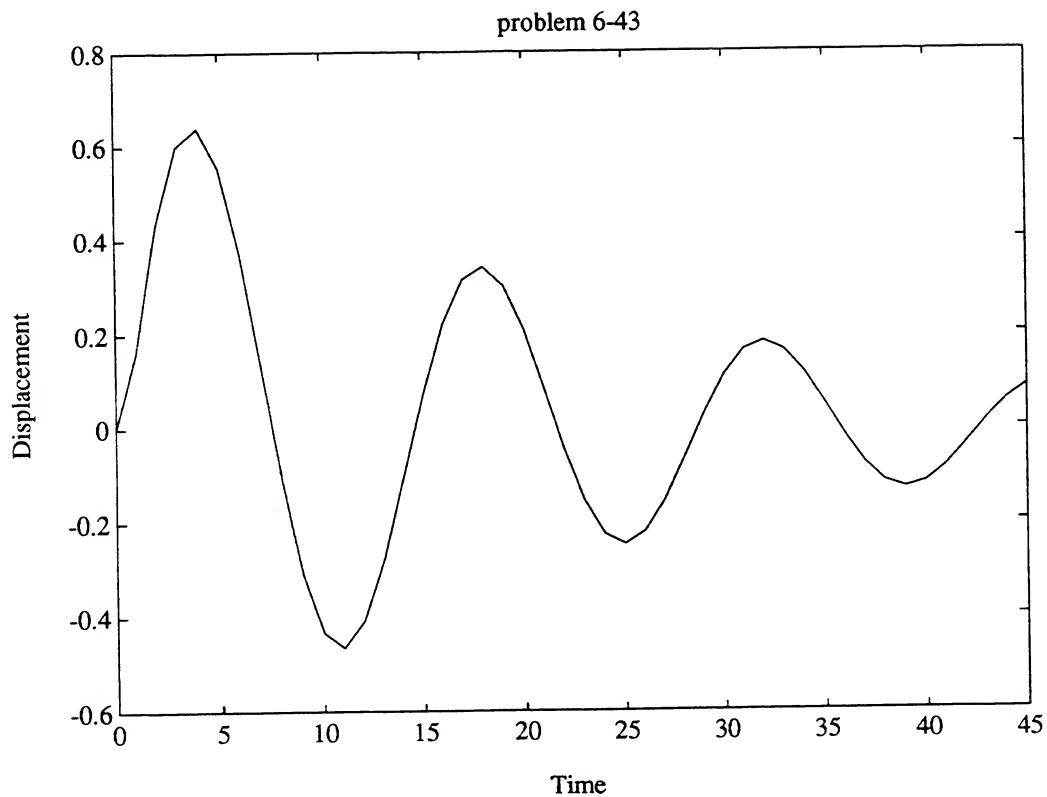
$$\text{New } A = Q^{-T} K Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix} \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m_3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m_1}(k_1+k_2) & -\frac{1}{\sqrt{m_1 m_2}}k_2 & 0 \\ -\frac{1}{\sqrt{m_1 m_2}}k_2 & \frac{1}{m_2}(k_2+k_3) & -\frac{1}{\sqrt{m_2 m_3}}k_3 \\ 0 & -\frac{1}{\sqrt{m_2 m_3}}k_3 & \frac{1}{m_3}k_3 \end{bmatrix} \therefore \text{symmetric} \\ \text{+ standard form.}$$

This procedure is valid for M a full symmetric matrix such as those encountered in the finite element formulation.

6-43

```
%This is the function file for problem 6-43
function [force]=f(t)
if t < 0.1
    force = 10*t;
elseif t < 0.3
    force=1-(1.7/.2)*(t-.1);
elseif t<0.5
    force=-.7+(1.2/.2)*(t-.3);
elseif t<0.7
    force=.5-4*(t-.5);
elseif t<0.9
    force=-.3+(5/2)*(t-.7);
elseif t<1
    force=.2-2*(t-.9);
else
    force=0;
end
```



6-44

Let θ_1, θ_2 and θ_3 be the angular rotations of the three pendulums with the positive direction is the same for all of them. Assume small oscillations

Let $\theta_1 = 1, \theta_2 = \theta_3 = 0 \Rightarrow T_1 = k + mgl, T_2 = -K, T_3 = 0$

Let $\theta_1 = \theta_3 = 0, \theta_2 = 1 \Rightarrow T_1 = -k, T_2 = 2k + mgl, T_3 = -k$

Let $\theta_1 = \theta_2 = 0, \theta_3 = 1 \Rightarrow T_1 = 0, T_2 = -K, T_3 = k + mgl$

$$\therefore \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} k+mgl & -k & 0 \\ -k & 2k+mgl & -k \\ 0 & -k & k+mgl \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \therefore [K] = \begin{bmatrix} k+mgl & -k & 0 \\ -k & 2k+mgl & -k \\ 0 & -k & k+mgl \end{bmatrix}$$

6-45

With linear dampers C_1 and C_2 the equation of motion is written as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} C_1 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

OR, $M\ddot{X} + C\dot{X} + KX = F$

Let $X = \tilde{P}Y$ and post multiply the D.E. by \tilde{P}^T

$$\tilde{P}^T M \tilde{P} \ddot{Y} + \tilde{P}^T C \tilde{P} \dot{Y} + \tilde{P}^T K \tilde{P} Y = \tilde{P}^T F \quad \dots (*)$$

$\tilde{P}^T M \tilde{P}$ and $\tilde{P}^T K \tilde{P}$ are diagonal matrices.

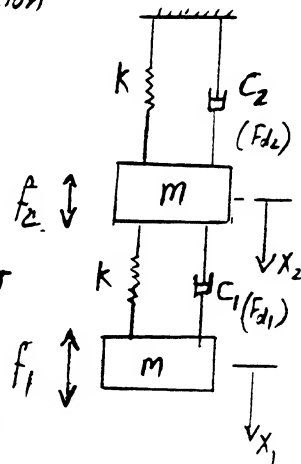
If $C = \alpha M + \beta K$, where α and β are constants (Rayleigh damping) then $\tilde{P}^T C \tilde{P}$ is diagonal and is equal to $\alpha I + \beta \Lambda$.

For example if $C_1 = C_2 = C$ then $\alpha = 0$ and $\beta = C/k$.

In this case (*) is written as

$$\ddot{y}_i + (\alpha + \beta \omega_i^2) \dot{y}_i + \omega_i^2 y_i = \tilde{f}_i(t), \quad i=1,2$$

Now, if the damping force is proportional to the square of the velocity, we have



$$F_{d1} = \pm c (\dot{x}_1 - \dot{x}_2)^2 \quad \text{and} \quad F_{d2} = c \dot{x}_2^2$$

opposite to the direction of motion.

Transforming \dot{x}_1 and \dot{x}_2 into \dot{y}_1 and \dot{y}_2 via \tilde{P} we express F_{d1} and F_{d2} in terms of \dot{y}_1 and \dot{y}_2 .

Answer: Yes, it is possible to develop an equivalent damping approach to this problem.

Choose f_1 and f_2 such that $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ are periodic. Find the energy dissipated per cycle (in steady state) for $i=1, 2$, i.e., W_{d1} and W_{d2} . By exciting the i th mode and letting the other one equal to zero plug in the steady state solution in the expressions for F_{d1} and F_{d2} in the y coordinates and find the dissipated energy per one cycle of the i th mode. (the sum of the energies dissipated by F_{d1} and F_{d2}), $i=1, 2$. Now, equate W_{d1} and W_{d2} with the corresponding energy dissipation of quadratic damping and find

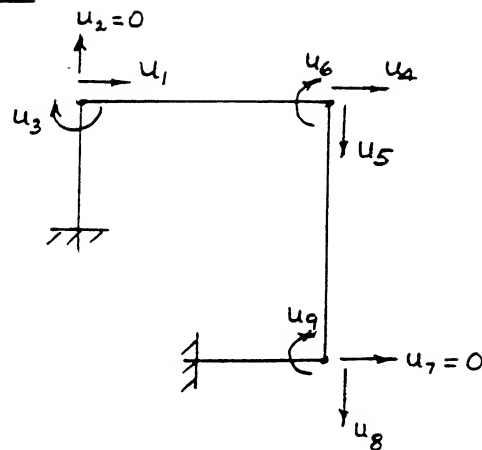
$$(\alpha + \beta \omega_i^2) c_f, \quad i=1, 2$$

$$\tilde{P}^T C_{eq} \tilde{P} = \begin{bmatrix} (\alpha + \beta \omega_1^2) c_f & 0 \\ 0 & (\alpha + \beta \omega_2^2) c_f \end{bmatrix}$$

C_{eq} can be found from this relation:

$$C_{eq} = \tilde{P}^{-T} \begin{bmatrix} (\alpha + \beta \omega_1^2) c_f & 0 \\ 0 & (\alpha + \beta \omega_2^2) c_f \end{bmatrix} \tilde{P}^{-1}$$

7-1



Disregard
 $u_2 = u_7 = 0$

Constraints are

$$u_1 = u_4$$

$$u_5 = u_8$$

There are 7 coordinates
 and 2 constraint eqs.

$$\therefore \text{DOF} = 5$$

7-2

7 Coordinates are $u_1, u_3, u_4, u_5, u_6, u_8, u_9$

4 of the above coordinates are associated with
 the two constraint equations, u_1, u_4, u_5, u_8 .

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_4 \\ u_5 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Above can be written as
 where u_4 & u_8 are chosen
 as gen. coords.

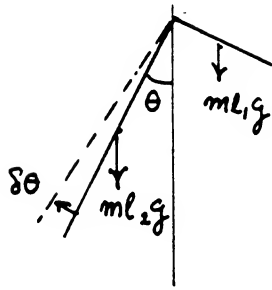
$$\begin{Bmatrix} u_1 \\ u_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_8 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_6 \\ u_8 \\ u_9 \end{Bmatrix}$$

gen. coords.

all coords.

7-3



$$-m l_2 g \frac{1}{2} l_2 [\cos \theta - \cos(\theta + \delta \theta)] + m l_1 g \frac{l_1}{2} [\sin(\theta + \delta \theta) - \sin \theta] = 0$$

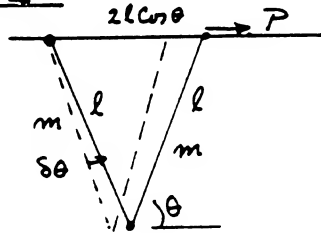
$$l_2^2 [\cos \theta - (\cos \theta \cos \delta \theta - \sin \theta \sin \delta \theta)] + l_1^2 [\sin \theta \cos \delta \theta + \cos \theta \sin \delta \theta] = 0$$

$$\cos \delta \theta \approx 1, \quad \sin \delta \theta \approx \delta \theta$$

$$l_2^2 [\sin \theta \cdot \delta \theta] + l_1^2 [\cos \theta \cdot \delta \theta] = 0$$

$$\therefore \tan \theta = \left(\frac{l_1}{l_2} \right)^2$$

7-4



$$U = 2 m g \frac{l}{2} \sin \theta$$

$$\delta U = m g l \cos \theta \delta \theta$$

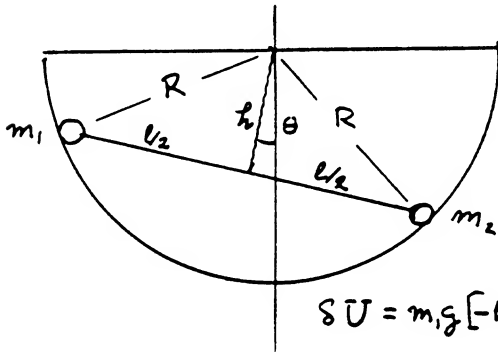
$$x = 2 l \cos \theta$$

$$\delta x = -2 l \sin \theta \delta \theta$$

$$\delta W = m g l \cos \theta \delta \theta - P 2 l \sin \theta \delta \theta = 0$$

$$\therefore \tan \theta = \frac{m g}{2 P}$$

7-5



$$h = \sqrt{R^2 - \left(\frac{l}{2} \right)^2}$$

$$2h = \sqrt{(2R)^2 - l^2}$$

$$U = m_1 g \left[h \cos \theta - \frac{l}{2} \sin \theta \right] + m_2 g \left[h \cos \theta + \frac{l}{2} \sin \theta \right]$$

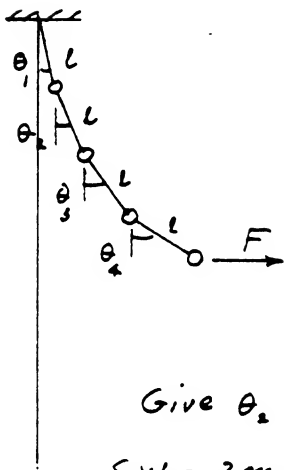
$$\delta U = m_1 g \left[-h \sin \theta \delta \theta - \frac{l}{2} \cos \theta \delta \theta \right]$$

$$+ m_2 g \left[-h \sin \theta \delta \theta + \frac{l}{2} \cos \theta \delta \theta \right] = 0$$

$$-(m_1 + m_2) h \sin \theta + (m_2 - m_1) \frac{l}{2} \cos \theta = 0$$

$$\tan \theta = \frac{(m_2 - m_1) \frac{l}{2}}{(m_1 + m_2) h} = \left(\frac{m_2 - m_1}{m_2 + m_1} \right) \frac{l}{\sqrt{(2R)^2 - l^2}}$$

7-6



Top mass position $x = l \sin \theta_1$
 $y = l \cos \theta_1$

Give θ_1 virtual displ. $\delta \theta_1$

$$\delta x = l \cos \theta_1 \delta \theta_1$$

$$\delta y = -l \sin \theta_1 \delta \theta_1$$

$$\delta W = 4mg(-l \sin \theta_1 \delta \theta_1) + Fl \cos \theta_1 \delta \theta_1 = 0$$

$$\therefore \tan \theta_1 = \frac{F}{4mg}$$

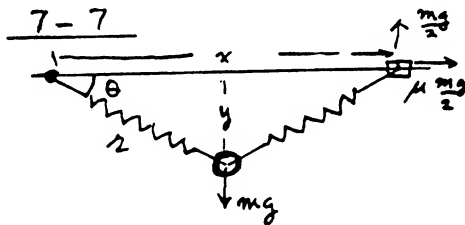
Give θ_2 virtual displ. $\delta \theta_2$

$$\delta W = 3mg(-l \sin \theta_2 \delta \theta_2) + Fl \cos \theta_2 \delta \theta_2 = 0$$

$$\therefore \tan \theta_2 = \frac{F}{3mg}$$

$$\text{etc. } \tan \theta_3 = \frac{F}{2mg}, \quad \tan \theta_4 = \frac{F}{mg}$$

7-7



$$y = r \sin \theta$$

$$\delta y = r \cos \theta \delta \theta$$

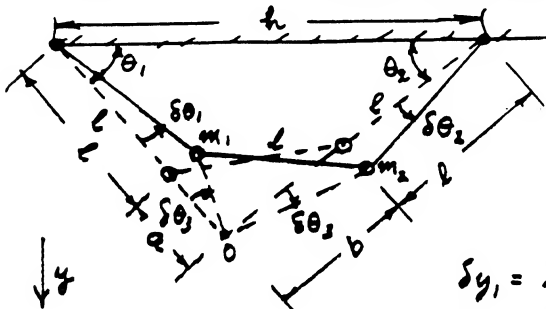
$$x = 2r \cos \theta$$

$$\delta x = -2r \sin \theta \delta \theta$$

$$\delta W = mg \delta y + \mu \frac{mg}{2} \delta x = 0 \quad \therefore \tan \theta = \frac{1}{\mu}$$

7-8

dotted lines show original position in equilibrium



$$\delta W = m_1 g \delta y_1 - m_2 g \delta y_2 = 0$$

The middle length rotates $\delta \theta_3$ about O.

$$l \delta \theta_1 = a \delta \theta_3$$

$$l \delta \theta_2 = b \delta \theta_3$$

$$\delta y_1 = 2 \delta \theta_1 \cos \theta_1 = a \delta \theta_3 \cos \theta_1$$

$$\delta y_2 = l \delta \theta_2 \cos \theta_2 = b \delta \theta_3 \cos \theta_2$$

7-8 Cont.

$$\therefore \delta W = (m_1 g a \cos \theta_1 - m_2 g b \cos \theta_2) \delta \theta_2 = 0$$

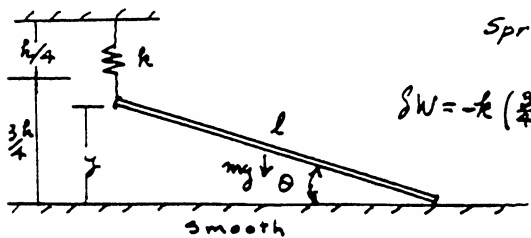
$$\left. \begin{aligned} \therefore m_1 a \cos \theta_1 - m_2 b \cos \theta_2 &= 0 \\ \text{Geometric Eqs.} \\ (l+a) \cos \theta_1 + (l+b) \cos \theta_2 &= h \\ l \sin \theta_1 - l \sin \theta_2 &= b \sin \theta_2 - a \sin \theta_1 \\ l^2 &= (a \cos \theta_1 + b \cos \theta_2)^2 + (b \sin \theta_2 - a \sin \theta_1)^2 \end{aligned} \right\} \begin{array}{l} 4 \text{ eqs} \\ 4 \text{ unknowns} \\ a, b, \theta_1, \theta_2 \end{array}$$

$$\sin(\theta_1 + \theta_2) = \frac{h}{l} \frac{(m_1 \cos \theta_1 \sin \theta_2 - m_2 \cos \theta_2 \sin \theta_1)}{(m_1 \cos \theta_1 - m_2 \cos \theta_2)}$$

7-9

$$\text{Stretch of spring} = \frac{3}{4}h - y$$

$$\text{Spring force} = k(\frac{3}{4}h - y)$$

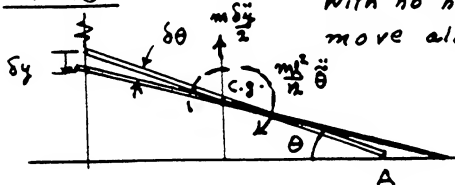


$$\delta W = -k(\frac{3}{4}h - y) \delta y + mg \frac{\delta y}{2} = 0$$

$$\therefore y = \frac{3}{4}h - \frac{mg}{2k}$$

$$\sin \theta = \frac{y}{l} = \frac{1}{l} \left(\frac{3h}{4} - \frac{mg}{2k} \right)$$

7-10



With no horizontal force on the bar, the c.g. must move along the vertical line.

$$\text{Let } \delta \theta = \tilde{\theta} \quad \therefore \theta = \theta_0 - \tilde{\theta}$$

$$\sin(\theta_0 - \tilde{\theta}) \approx \sin \theta_0 - \tilde{\theta} \cos \theta_0$$

$$\cos(\theta_0 - \tilde{\theta}) \approx \cos \theta_0 + \tilde{\theta} \sin \theta_0$$

$$\text{Forces are balanced at } \theta_0. \quad \delta y = l \sin \theta_0 - l \sin(\theta_0 - \tilde{\theta}) = (l \cos \theta_0) \tilde{\theta}$$

$$\text{Spring force due to } \tilde{\theta} \text{ is } -k \delta y = -(k l \cos \theta_0) \tilde{\theta}$$

$$\delta W = (-k l \cos \theta_0) \tilde{\theta} \cdot (l \cos \theta_0 \tilde{\theta}) - \left(\frac{m}{2} l \cos \theta_0 \tilde{\theta} \right) \left(\frac{l \cos \theta_0 \tilde{\theta}}{2} \right) - \left(\frac{m l^2}{12} \ddot{\tilde{\theta}} \right) \tilde{\theta} = 0$$

$$\left(\frac{m l^2}{12} + \frac{m l^2}{4} \cos^2 \theta_0 \right) \ddot{\tilde{\theta}} + (k l^2 \cos^2 \theta_0) \tilde{\theta} = 0$$

For small θ_0 above eq. reduces to

$$\left(\frac{m l^2}{3} \right) \ddot{\tilde{\theta}} + (k l^2) \tilde{\theta} = 0$$

7-11

$$\left[(m l_1) \frac{l_1^2}{3} + (m l_2) \frac{l_2^2}{3} \right] \ddot{\theta} = -(m l_2 g) \frac{l_2}{2} \sin \theta + (m l_1 g) \frac{l_1}{2} \cos \theta$$

Let $\theta = \theta_0 + \theta_{\sim}$

$$\sin \theta \approx \sin \theta_0 + \theta_{\sim} \cos \theta_0$$

$$\cos \theta \approx \cos \theta_0 - \theta_{\sim} \sin \theta_0$$

but $-(m l_2 g) \frac{l_2}{2} \sin \theta_0 + (m l_1 g) \frac{l_1}{2} \cos \theta_0 = 0$

$$\therefore \left(m \frac{l_1^3}{3} + m \frac{l_2^3}{3} \right) \ddot{\theta}_{\sim} = -\frac{m g}{2} (l_2^2 \cos \theta_0 + l_1^2 \sin \theta_0) \theta_{\sim}$$

$$\ddot{\theta}_{\sim} + \frac{3}{2} g \left(\frac{l_2^2 \cos \theta_0 + l_1^2 \sin \theta_0}{l_1^3 + l_2^3} \right) \theta_{\sim} = 0$$

where $\tan \theta_0 = \left(\frac{l_1}{l_2} \right)^2$

7-12

$$T = \frac{1}{2} (m_1 + m_2) R^2 \dot{\theta}^2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = (m_1 + m_2) R^2 \ddot{\theta}$$

$$h = \sqrt{R^2 + \left(\frac{l}{2} \right)^2}$$

$$U = -(m_1 + m_2) g h \cos \theta - (m_2 - m_1) g \frac{l}{2} \sin \theta$$

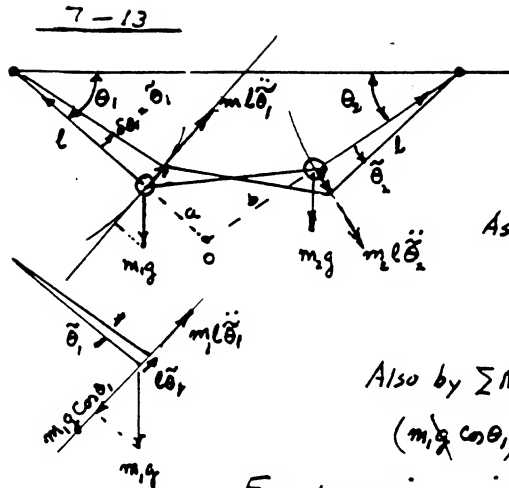
$$\frac{\partial U}{\partial \theta} = (m_1 + m_2) g h \sin \theta - (m_2 - m_1) g \frac{l}{2} \cos \theta$$

Lagranges Eq.

$$(m_1 + m_2) R^2 \ddot{\theta} + (m_1 + m_2) g h \sin \theta - (m_2 - m_1) g \frac{l}{2} \cos \theta = 0$$

Let $\theta = \theta_0 + \theta_{\sim}$

$$(m_1 + m_2) R^2 \ddot{\theta}_{\sim} + [(m_1 + m_2) g h \cos \theta_0 + (m_2 - m_1) g \frac{l}{2} \sin \theta_0] \theta_{\sim} = 0$$



Motion of each mass is along circle of radius l . Virtual displ. $= l \tilde{\theta}_1$ and $l \tilde{\theta}_2$

where $\theta_1 = \bar{\theta}_1 + \tilde{\theta}_1$ and $\theta_2 = \bar{\theta}_2 + \tilde{\theta}_2$

As in Prob 7-8

$$\delta \theta_1 = \tilde{\theta}_1 = \frac{a}{l} \delta \theta_2 = \frac{a}{l} \tilde{\theta}_2$$

$$\delta \theta_2 = \tilde{\theta}_2 = \frac{b}{l} \delta \theta_1 = \frac{b}{l} \tilde{\theta}_1 = \frac{b}{a} \tilde{\theta}_1$$

Also by ΣM_o

$$(m_1 g \cos \theta_1) a = (m_2 g \cos \theta_2) b \quad \text{same eq. as Prob 7-8}$$

For dynamics include inertia forces $m_1 l \ddot{\theta}_1$ & $m_2 l \ddot{\theta}_2$ in tangential direction $l \tilde{\theta}_1$ and $l \tilde{\theta}_2$. Then work done by virtual displ. $l \tilde{\theta}_1$ and $l \tilde{\theta}_2$ is

$$\delta W = (m_1 l \ddot{\theta}_1 - m_1 g \cos \theta_1) l \tilde{\theta}_1 + (m_2 l \ddot{\theta}_2 + m_2 g \cos \theta_2) l \tilde{\theta}_2 = 0$$

For small $\tilde{\theta}$ with $\theta = \bar{\theta} \pm \tilde{\theta}$ $\cos \theta_1 = \cos \bar{\theta}_1 - \tilde{\theta}_1 \sin \bar{\theta}_1$

$$\cos \theta_2 = \cos \bar{\theta}_2 + \tilde{\theta}_2 \sin \bar{\theta}_2$$

$$\tilde{\theta}_2 = \frac{b}{a} \tilde{\theta}_1 = \frac{m_1}{m_2} \frac{\cos \bar{\theta}_1}{\cos \bar{\theta}_2} \tilde{\theta}_1$$

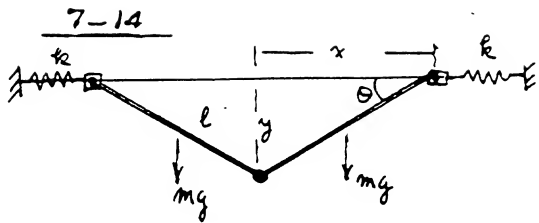
$$+ [m_1 l + m_2 l \left(\frac{b}{a}\right)^2] \ddot{\theta}_1 \tilde{\theta}_1 - g \left\{ m_1 (\cos \bar{\theta}_1 - \tilde{\theta}_1 \sin \bar{\theta}_1) \tilde{\theta}_1 - m_2 \left(\cos \bar{\theta}_2 + \frac{b}{a} \tilde{\theta}_1 \sin \bar{\theta}_2 \right) \frac{b}{a} \tilde{\theta}_1 \right\} = 0$$

Since $m_1 \cos \bar{\theta}_1 \cdot a - m_2 \cos \bar{\theta}_2 \cdot b = 0$ from Prob. 8-6 & above

$$\left[m_1 l + m_2 l \left(\frac{b}{a}\right)^2 \right] \ddot{\theta}_1 \tilde{\theta}_1 + g \left\{ m_1 \sin \bar{\theta}_1 + m_2 \left(\frac{m_1 \cos \bar{\theta}_1}{m_2 \cos \bar{\theta}_2} \right)^2 \right\} \tilde{\theta}_1 \tilde{\theta}_1 = 0$$

$$\left[m_1 + m_2 \left(\frac{m_1 \cos \bar{\theta}_1}{m_2 \cos \bar{\theta}_2} \right)^2 \right] \ddot{\theta}_1 + \frac{g}{l} \left[m_1 \sin \bar{\theta}_1 + m_2 \left(\frac{m_1 \cos \bar{\theta}_1}{m_2 \cos \bar{\theta}_2} \right)^2 \right] \tilde{\theta}_1 = 0$$

$$\therefore \omega_n^2 = \frac{g}{l} \cdot \frac{m_1 \sin \bar{\theta}_1 + m_2 \left(\frac{m_1 \cos \bar{\theta}_1}{m_2 \cos \bar{\theta}_2} \right)^2}{m_1 + m_2 \left(\frac{m_1 \cos \bar{\theta}_1}{m_2 \cos \bar{\theta}_2} \right)^2}$$



Let $\theta_0 = \text{equilib angle}$
 Spring force = F_{s0} at $\theta = \theta_0$

$$x = l \cos \theta \quad \delta x = -l \sin \theta_0 \delta \theta$$

$$y = l \sin \theta \quad \delta y = l \cos \theta_0 \delta \theta$$

$$\delta W = 2mg \frac{\delta y}{2} + 2F_{s0} \delta x = 0$$

$$(mg \cos \theta_0 - 2F_{s0} \sin \theta_0) l \delta \theta = 0$$

If $F_s = 0$ at $\theta = 0$ then $F_{s0} = k(l-x) = k l (1 - \cos \theta_0)$

$$\tan \theta_0 = \frac{mg}{2kl(1 - \cos \theta_0)}$$

solve by trial for given
 value of $\frac{mg}{2kl}$

7-15

$$T = \frac{1}{2} m_0 [(\dot{r})^2 + \dot{\theta}^2] + \frac{1}{2} \left(m \frac{l^2}{3} \right) \dot{\theta}^2$$

$$U = \frac{1}{2} k (r - r_0)^2 - m_0 g r \cos \theta - mg \frac{l}{2} \cos \theta$$

$$\frac{\partial T}{\partial \dot{\theta}} = m_0 \dot{r}^2 \dot{\theta} + m \frac{l^2}{3} \dot{\theta} \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial U}{\partial \theta} = m_0 g r \sin \theta + mg \frac{l}{2} \sin \theta$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0$$

$$m_0 [r \ddot{\theta} + 2 \dot{r} \dot{\theta}] + \left(m_0 g r + mg \frac{l}{2} \right) \sin \theta = 0$$

$$\frac{\partial T}{\partial \dot{r}} = m_0 \dot{r} \quad \frac{\partial T}{\partial r} = m_0 r \dot{\theta}^2$$

$$\frac{\partial U}{\partial r} = -m_0 g \cos \theta + k(r - r_0)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} + \frac{\partial U}{\partial r} = 0$$

$$m_0 \ddot{r} - m_0 r \dot{\theta}^2 + k(r - r_0) - m_0 g \cos \theta = 0$$

7-16

$$T = \frac{1}{2} \int_0^l m \dot{y}^2 dx$$

$$U = \frac{1}{2} k y^2(l) + \frac{1}{2} K y'^2(0) + \frac{1}{2} \int_0^l EI \left(\frac{d^2 y}{dx^2} \right)^2 dx$$

$$y = \frac{x}{l} q_1 + q_2 \sin \frac{\pi x}{l}$$

$$\dot{y}^2 = \left(\frac{x}{l} \right)^2 \dot{q}_1^2 + 2 \left(\frac{x}{l} \right) \dot{q}_1 \dot{q}_2 \sin \frac{\pi x}{l} + \dot{q}_2^2 \sin^2 \frac{\pi x}{l}$$

$$y' = \frac{1}{l} q_1 + q_2 \frac{\pi}{l} \cos \frac{\pi x}{l}$$

$$y'^2 = \frac{1}{l^2} q_1^2 + \frac{2}{l} q_1 q_2 \frac{\pi}{l} \cos \frac{\pi x}{l} + q_2^2 \left(\frac{\pi}{l} \right)^2 \cos^2 \frac{\pi x}{l}$$

$$y'' = -q_2 \left(\frac{\pi}{l} \right)^2 \sin \frac{\pi x}{l}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} = \ddot{q}_1 \int_0^l m \left(\frac{x}{l} \right)^2 dx + \ddot{q}_2 \int_0^l \frac{x}{l} \sin \frac{\pi x}{l} dx = \frac{ml}{3} \ddot{q}_1 + \frac{l}{\pi} \ddot{q}_2$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} = \ddot{q}_1 \int_0^l \frac{x}{l} \sin \frac{\pi x}{l} dx + \ddot{q}_2 \int_0^l \sin^2 \frac{\pi x}{l} dx = \frac{l}{\pi} \ddot{q}_1 + \frac{l}{2} \ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = k q_1 + \frac{K}{l^2} q_1 + \frac{K \pi}{l^2} q_2$$

$$\frac{\partial U}{\partial q_2} = \frac{K \pi}{l^2} q_1 + K \left(\frac{\pi}{l} \right)^2 q_2 + EI \left(\frac{\pi}{l} \right)^4 \frac{l}{2} q_2$$

$$ml \begin{bmatrix} \frac{1}{3} & \frac{1}{2\pi} \\ \frac{1}{2\pi} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \frac{EI}{l^3} \begin{bmatrix} (k + \frac{K}{l^2}) \frac{l^3}{EI} & -\frac{\pi K}{l^2} \frac{l^3}{EI} \\ -\frac{\pi K}{l^2} \frac{l^3}{EI} & (\frac{\pi^4}{2} + \frac{\pi^2 K}{l^2} \frac{l^3}{EI}) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

7-17

$$T = \frac{1}{2} \left(\frac{m\ell^2}{3} \right) \dot{\theta}_1^2 + \frac{1}{2} \left(\frac{m\ell^2}{12} \right) \dot{\theta}_2^2 + \frac{1}{2} m (\ell \dot{\theta}_1 + \frac{\ell}{2} \dot{\theta}_2)^2$$

$$U = \frac{1}{2} k \left(\frac{\ell}{2} \theta_1 \right)^2 + mg \frac{\ell}{2} (1 - \cos \theta_1) + mg \left[\ell (1 - \cos \theta_1) + \frac{\ell}{2} (1 - \cos \theta_2) \right]$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} = \left(\frac{m\ell^2}{3} \right) \ddot{\theta}_1 + m (\ell \ddot{\theta}_1 + \frac{\ell}{2} \ddot{\theta}_2) \ell$$

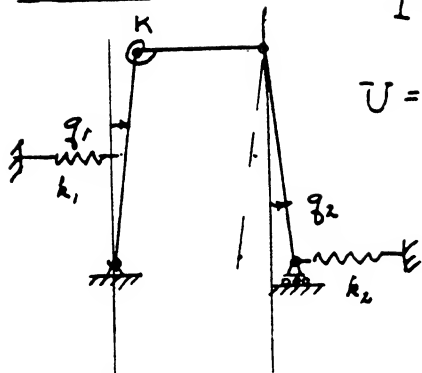
$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} = \left(\frac{m\ell^2}{12} \right) \ddot{\theta}_2 + m (\ell \ddot{\theta}_1 + \frac{\ell}{2} \ddot{\theta}_2) \frac{\ell}{2}$$

$$\frac{\partial U}{\partial \theta_1} = k \left(\frac{\ell}{2} \theta_1 \right) + mg \frac{\ell}{2} \sin \theta_1 + mg \ell \sin \theta_1$$

$$\frac{\partial U}{\partial \theta_2} = mg \frac{\ell}{2} \sin \theta_2$$

$$\begin{bmatrix} \left(\frac{m\ell^2}{3} + m\ell^2 \right) & m\frac{\ell^2}{2} \\ m\frac{\ell^2}{2} & \left(\frac{m\ell^2}{12} + \frac{m\ell^2}{4} \right) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} \left(k\frac{\ell}{2} + \frac{3}{2}mg\ell \right) & 0 \\ 0 & mg\frac{\ell}{2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

7-18



$$T = \frac{1}{2}(m_1 + m_2)(2l\dot{q}_1)^2 + \frac{1}{2}J_1\dot{q}_1^2 + \frac{1}{2}J_2\dot{q}_2^2$$

$$U = \frac{1}{2}k_1(lq_1)^2 + \frac{1}{2}Kq_1^2 + \frac{1}{2}k_2(2l)^2(q_1 + q_2)^2$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = (m_1 + m_2)(4l^2)\ddot{q}_1 + J_1\ddot{q}_1$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = J_2\ddot{q}_2$$

$$\frac{\partial U}{\partial q_1} = l^2 k_1 q_1 + K q_1 + k_2 (4l^2)(q_1 + q_2)$$

$$\frac{\partial U}{\partial q_2} = 4l^2 k_2 (q_1 + q_2)$$

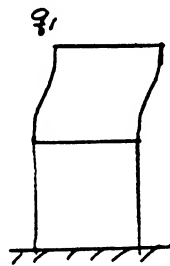
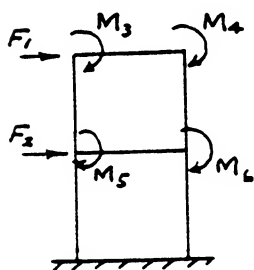
$$[(m_1 + m_2)4l^2 + J_1]\ddot{q}_1 + [l^2 k_1 + K + 4l^2 k_2]q_1 + 4l^2 k_2 q_2 = 0$$

$$J_2\ddot{q}_2 + 4l^2 k_2 (q_1 + q_2) = 0$$

7-19

Refer to Fig 7.1-4.

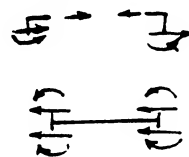
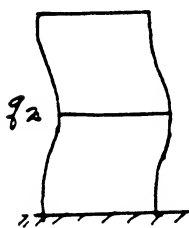
Assume $l_1 = l_2 = l$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 & 0 & 0 \\ -6l & 0 & 0 & 0 & 0 & 0 \\ -6l & 0 & 0 & 0 & 0 & 0 \\ -6l & 0 & 0 & 0 & 0 & 0 \\ -6l & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

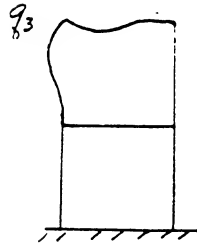
Examine FBD of each of the four corners for above

Refer to Fig 6.4-2



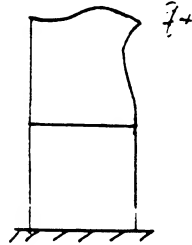
$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 0 & -24 & 0 & 0 & 0 & 0 \\ 0 & 48 & 0 & 0 & 0 & 0 \\ 6l & 0 & 0 & 0 & 0 & 0 \\ 6l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ q_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

7-19 Cont.

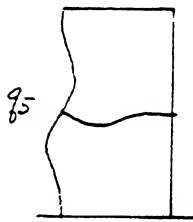


Examine FBD
of each corner

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 0 & 0 & -6l & 0 & 0 & 0 \\ \vdots & \vdots & 6l & \vdots & \vdots & \vdots \\ \vdots & \vdots & 8l^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & 2l^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & 2l^2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 0 & 0 & 0 & -6l & 0 & 0 \\ \vdots & \vdots & \vdots & 6l & \vdots & \vdots \\ \vdots & \vdots & \vdots & 2l^2 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 8l^2 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 2l^2 & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 0 & 0 & 0 & 0 & -6l & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 2l^2 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 12l^2 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 12l^2 & \vdots \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_5 \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -6l \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2l^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2l^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 12l^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_6 \end{Bmatrix}$$

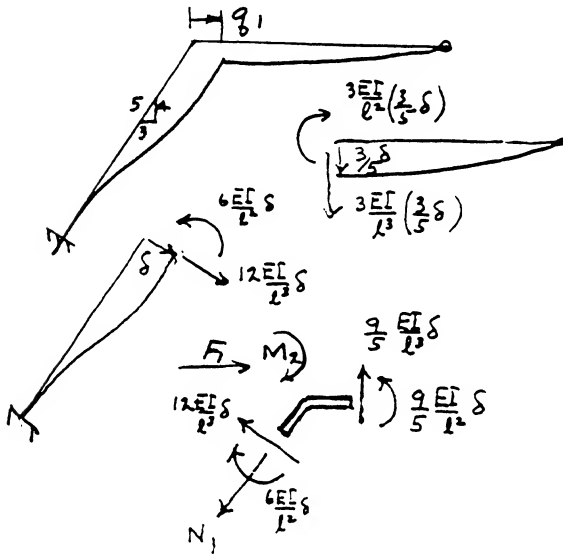
$$[k] = \frac{EI}{l^3} \begin{bmatrix} 24 & -24 & -6l & -6l & -6l & -6l \\ -24 & 48 & 6l & 6l & 0 & 0 \\ -6l & 6l & 8l^2 & 2l^2 & 2l^2 & 0 \\ -6l & 6l & 2l^2 & 8l^2 & 0 & 2l^2 \\ -6l & 0 & 2l^2 & 0 & 12l^2 & 2l^2 \\ -6l & 0 & 0 & 2l^2 & 2l^2 & 12l^2 \end{bmatrix}$$

7-19 Cont.

$$[m] = \begin{bmatrix} 2m & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 & 0 & 0 \\ 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & 0 & J \end{bmatrix}$$

$$[m]\{\ddot{q}\} + [k]\{q\} = 0$$

7-20



$$\sum F_y = -\frac{4}{5} N_1 + \left(\frac{9}{5} + \frac{3}{5} \times 12\right) \frac{EI}{l^3} \delta = 0$$

$$\therefore N_1 = \frac{45}{4} \frac{EI}{l^3} \delta$$

$$\sum F_x = -F_1 + \left[\frac{3}{5} \left(\frac{45}{4}\right) + \frac{4}{5} \times 12\right] \frac{EI}{l^3} \delta = 0$$

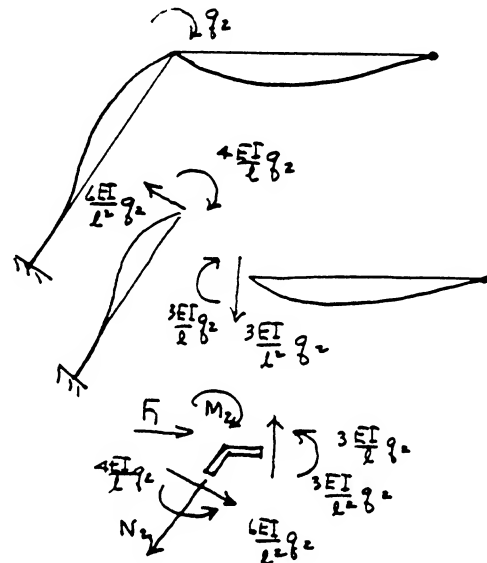
$$F_1 = 16.35 \frac{EI}{l^3} \delta$$

$$\sum M \quad M_2 = \left(\frac{9}{5} - 6\right) \frac{EI}{l^2} \delta = -4.20 \frac{EI}{l^2} \delta$$

$$\text{but } \delta = \frac{5}{4} q_1$$

$$\therefore F_1 = 20.44 \frac{EI}{l^3} q_1$$

$$M_2 = -5.25 \frac{EI}{l^2} q_1$$



$$\sum F_y = 3 \frac{EI}{l^2} q_2 - \frac{3}{5} \left(6 \frac{EI}{l^2}\right) q_2 - \frac{4}{5} N_2 = 0$$

$$\therefore N_2 = -0.75 \frac{EI}{l^2} q_2$$

$$\sum F_x = 0 \quad \text{gives}$$

$$F_1 = \left[\frac{3}{5} (-0.75) - 6 \times \frac{4}{5}\right] \frac{EI}{l^2} q_2$$

$$F_1 = -5.25 \frac{EI}{l^2} q_2$$

$$M_2 = 7 \frac{EI}{l^2} q_2$$

7-20 Cont.

$$\begin{Bmatrix} F_1 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 20.44 & -5.25l \\ -5.25l & 7.0l^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

7-21 From Fig 6.4-20a rotation of right end is
 due to q_1 , $\frac{3}{2} \left(\frac{8}{5} \frac{5}{l} \right) = \frac{3}{2} \left(\frac{3}{5} \times \frac{5}{4} \frac{q_1}{l} \right) = \frac{9}{8} \frac{q_1}{l}$

due to q_2 , $\frac{1}{2} q_2$

$$T = \frac{1}{2} (m_1 + m_2) \dot{q}_1^2 + \frac{1}{2} J_1 \dot{q}_2^2 + \frac{1}{2} J_2 \left(\frac{9}{8} \frac{\dot{q}_1}{l} + \frac{1}{2} \dot{q}_2 \right)^2$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} = (m_1 + m_2) \ddot{q}_1 + J_2 \left(\frac{9}{8l} \ddot{q}_1 + \frac{1}{2} \ddot{q}_2 \right) \frac{9}{8l}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} = J_1 \ddot{q}_2 + J_2 \left(\frac{9}{8l} \dot{q}_1 + \frac{1}{2} \dot{q}_2 \right) \frac{1}{2}$$

$$U = \frac{1}{2} (q_1, q_2) \begin{bmatrix} k \\ k_0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \frac{1}{2} k_0 q_1^2 + \frac{1}{2} K_0 \left(\frac{9}{8l} q_1 + \frac{1}{2} q_2 \right)^2$$

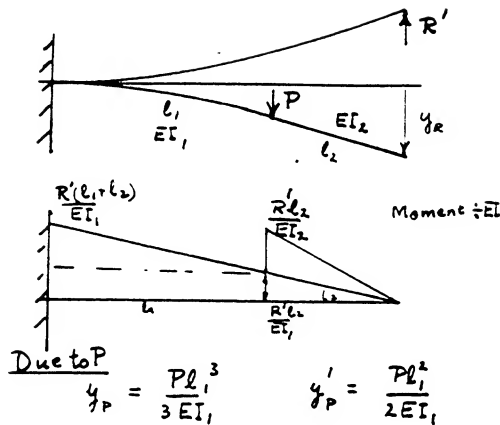
$$\frac{\partial U}{\partial q_{1,2}} = \begin{bmatrix} \left\{ 20.44 \frac{EI}{l^3} + k_0 + \left(\frac{9}{8l} \right)^2 K_0 \right\} & \left\{ -5.25 \frac{EI}{l^2} + \frac{1}{2} \times \frac{9}{8l} K_0 \right\} \\ \left\{ -5.25 \frac{EI}{l^2} + \frac{1}{2} \times \frac{9}{8l} K_0 \right\} & \left\{ 7.0 \frac{EI}{l} + \frac{1}{4} K_0 \right\} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Eq. of motion

$$\begin{bmatrix} \left\{ m_1 + m_2 + \left(\frac{9}{8l} \right)^2 J_2 \right\} & \left\{ \frac{1}{2} \times \frac{9}{8l} J_2 \right\} \\ \left\{ \frac{1}{2} \times \frac{9}{8l} J_2 \right\} & \left\{ J_1 + \frac{1}{4} J_2 \right\} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \left\{ 20.44 \frac{EI}{l^3} + k_0 + \left(\frac{9}{8l} \right)^2 K_0 \right\} & \left\{ -5.25 \frac{EI}{l^2} + \frac{9}{16l} K_0 \right\} \\ \left\{ -5.25 \frac{EI}{l^2} + \frac{9}{16l} K_0 \right\} & \left\{ 7.0 \frac{EI}{l} + \frac{1}{4} K_0 \right\} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assign numbers for $m_1, m_2, J_1, J_2, k_0, K_0$ & l
 for normal mode determination.

7-22



$$\therefore y_R = \frac{Pl_1^3}{3EI_1} + \left(\frac{Pl_1^2}{2EI_1} \right) l_2$$

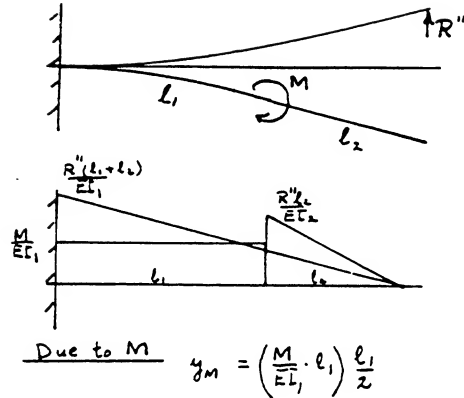
Due to R'

$$y_R = \left(\frac{1}{2} \frac{R'l_2}{EI_2} \cdot l_2 \right) \frac{2}{3} l_2 + \left(\frac{R'l_1}{EI_1} \cdot l_1 \right) \left(l_2 + \frac{1}{2} l_1 \right) + \left(\frac{1}{2} \frac{R'l_1}{EI_1} \cdot l_1 \right) \left(l_2 + \frac{2}{3} l_1 \right)$$

Equate y_R due to P and due to R' (use $\frac{EI_1}{EI_2} = 2$)

$$\frac{P}{EI_1} \left[\frac{l_1^3}{3} + \frac{l_1^2 l_2}{2} \right] = \frac{R'}{EI_1} \left[\frac{2}{3} l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3} \right]$$

$$= R' [b]$$



$$y'_M = \frac{M}{EI_1} l_1$$

$$y_R = \frac{M}{EI_1} \frac{l_1^2}{2} + \frac{M l_1 l_2}{EI_1}$$

Due to R''

y_R same as that for R' with R' replaced by R''

Equate y_R due to M and due to R''

$$\frac{M}{EI_1} \left(\frac{l_1^2}{2} + l_1 l_2 \right) = \frac{R''}{EI_1} \left[\frac{2}{3} l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3} \right]$$

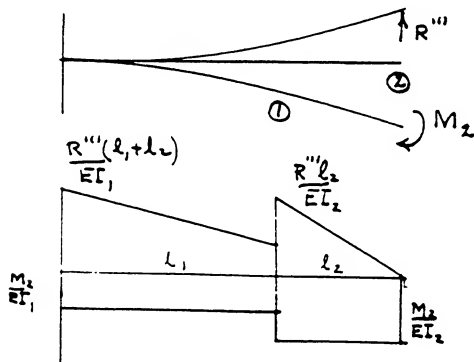
$$= R'' [b]$$

$$M_1 = -M - Pl_1 + R(l_1 + l_2) \quad \text{where } R = R' + R''$$

$$R = \frac{P \left(\frac{1}{3} l_1^3 + \frac{1}{2} l_1^2 l_2 \right) + M \left(\frac{1}{2} l_1^2 + l_1 l_2 \right)}{\frac{2}{3} l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{1}{3} l_1^3}$$

7-23

Need stiffness matrix for M, P and M2 at end of beam



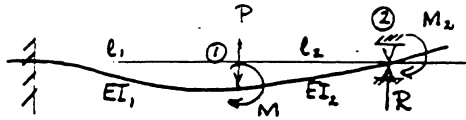
$$y_R = \frac{M_2 l_1}{EI_1} \left(l_2 + \frac{l_1}{2} \right) + \frac{M_2}{EI_2} \frac{l_2^2}{2}$$

$$y_R = \frac{R'''}{EI_1} \left[\frac{2}{3} l_2^3 + l_1 l_2^2 + l_2 l_1^2 + \frac{l_1^3}{3} \right]$$

Equate

$$\frac{M_2}{EI_1} \left[l_1 l_2 + \frac{l_1^2}{2} + 2 \cdot \frac{l_1^2}{2} \right] = \frac{R'''}{EI_1} [b]$$

7-23 Cont.



Determine y_1 , θ_1 , y_2 , θ_2

Set $y_2 = 0$ to find R

Determine flexibility matrix & invert.
 $EI_1 = 2 EI_2$

$$y_1 = \frac{Pl_1^3}{8EI_1} + \frac{Ml_1^2}{2EI_1} + \frac{M_2l_1^2}{2EI_1} - \frac{1}{2} \left[\frac{R(l_1+l_2)l_1}{EI_1} - \frac{Rl_1l_2}{EI_1} \right] \frac{2}{3}l_1 - \frac{Rl_2}{EI_1} \frac{l_1^2}{2}$$

$$\theta_1 = \frac{Pl_1^2}{2EI_1} + \frac{Ml_1}{EI_1} + \frac{M_2l_1}{EI_1} - \frac{1}{2} \left[\frac{R(l_1+l_2)}{EI_1} + \frac{Rl_2}{EI_1} \right] l_1$$

$$y_2 = \frac{1}{2} \frac{Pl_1^2}{EI_1} \left(\frac{2}{3}l_1 + l_2 \right) + \frac{Ml_1}{EI_1} \left(\frac{1}{2}l_1 + l_2 \right) + \frac{M_2l_1}{EI_1} \left(\frac{l_1}{2} + l_2 \right) + \frac{M_2}{EI_2} \frac{l_2^2}{2} - \frac{1}{2} \frac{R(l_1+l_2)}{EI_1} (l_1+l_2) \frac{2}{3}(l_1+l_2) - \frac{1}{2} \left[\frac{Rl_1^2}{EI_1} - \frac{Rl_1l_2}{EI_1} \right] \frac{2}{3}l_2$$

$$\theta_2 = \frac{Pl_1^2}{2EI_1} + \frac{Ml_1}{EI_1} + \frac{M_2l_1}{EI_1} + \frac{M_2l_2}{EI_2} - \frac{1}{2} \frac{R(l_1+l_2)^2}{EI_1} - \frac{1}{2} \left(\frac{Rl_1}{EI_1} - \frac{Rl_2}{EI_1} \right) l_2$$

To simplify algebra let $l_1 = l_2 = l$ & $EI_1 = 2EI_2$, result is

$$\begin{Bmatrix} y_1 \\ \theta_1 \\ \theta_2 \\ y_2=0 \end{Bmatrix} = \frac{1}{EI_1} \begin{bmatrix} \frac{l^3}{3} & \frac{l^2}{2} & \frac{l^2}{2} & -\frac{5}{6}l^3 \\ \frac{l^2}{2} & l & l & -\frac{3}{2}l^2 \\ \frac{l^2}{2} & l & 3l & -\frac{5}{2}l^2 \\ \frac{5}{6}l^3 & \frac{3}{2}l^2 & \frac{5}{2}l^2 & -3l^3 \end{bmatrix} \begin{Bmatrix} P \\ M \\ M_2 \\ R \end{Bmatrix}$$

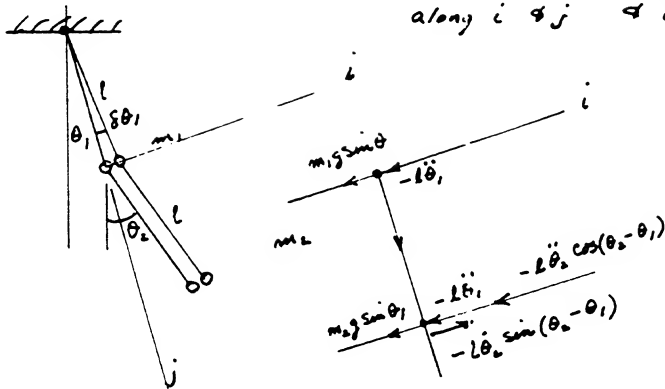
from $y_2=0$ $R = \frac{5}{18}P + \frac{1}{2}\frac{M}{l} + \frac{5}{6}\frac{M_2}{l}$ Subst. into above to obtain

$$\begin{Bmatrix} y_1 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{1}{EI_1} \begin{bmatrix} 0.1018l^3 & 0.0833l^2 & -0.1944l^2 \\ 0.0833l^2 & 0.25l & -0.25l \\ -0.1944l^2 & -0.25l & 0.0916l \end{bmatrix} \begin{Bmatrix} P \\ M \\ M_2 \end{Bmatrix}$$

invert by computer program to obtain stiffness matrix
Let numeric value of $l=1.0$

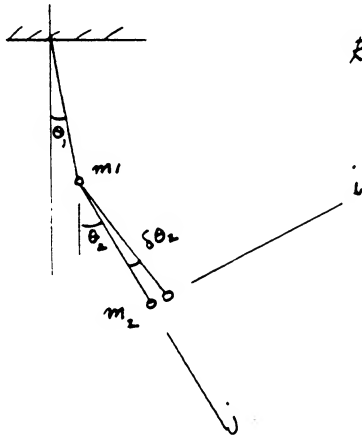
I-24

Resolve all forces including inertia forces
along i & j & dot product with $l\delta\theta_1 i$



$$\sum (F - m\ddot{r}) \cdot l\delta\theta_1 i = -(m_1 + m_2)g \sin\theta_1 - (m_1 + m_2)l\ddot{\theta}_1 - m_2 l \ddot{\theta}_2 \cos(\theta_2 - \theta_1) + m_2 l \ddot{\theta}_2^2 \sin(\theta_2 - \theta_1) = 0$$

$$\text{or } (m_1 + m_2)l\ddot{\theta}_1 + m_2 l \ddot{\theta}_2 \cos(\theta_2 - \theta_1) - m_2 l \ddot{\theta}_2^2 \sin(\theta_2 - \theta_1) + (m_1 + m_2)g \sin\theta_1 = 0$$



Resolve all forces along new i j
 \perp & \parallel to l_2 and dot with $l\delta\theta_2 i$

$$m_2 l \ddot{\theta}_2 + m_2 l \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 l \ddot{\theta}_1^2 \sin(\theta_2 - \theta_1) + m_2 g \sin\theta_2 = 0$$

Try checking these equations from Lagrange's eqs.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2} = 0$$

7-25

$$T = \frac{1}{2} (3m) \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2$$

$$U = \frac{1}{2} k q_1^2 + \frac{1}{2} k (q_2 - q_1)^2 + \frac{1}{2} k q_2^2$$

$$L = T - U = \frac{1}{2} (3m) \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 - \frac{1}{2} k q_1^2 - \frac{1}{2} k (q_2 - q_1)^2 - \frac{1}{2} k q_2^2$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_1} \right) = 3m \ddot{q}_1, \quad \frac{\partial L}{\partial q_1} = -k q_1 + k (q_2 - q_1)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_2} \right) = m \ddot{q}_2, \quad \frac{\partial L}{\partial q_2} = -k (q_2 - q_1) - k q_2$$

Eq. in matrix form

$$m \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

7-26

Assume that the table can move only in the vertical direction and the mass m is rotating with angular speed ω around the vertical axis. Let r be the distance between m and the axis of rotation. Denote the vertical displacement by x .

$$\text{Kinetic energy } T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (r\omega)^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \omega^2$$

$$\text{Potential energy } U = \frac{1}{2} (4k) x^2 = 2kx^2$$

Let θ be the angular displacement of m . Therefore, $\omega = d\theta/dt = \dot{\theta}$.

We use Lagrange's equations to find the differential equations of motion.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0 \Rightarrow m\ddot{x} + 4kx = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0 \Rightarrow \ddot{\theta} = 0$$

In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 4k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

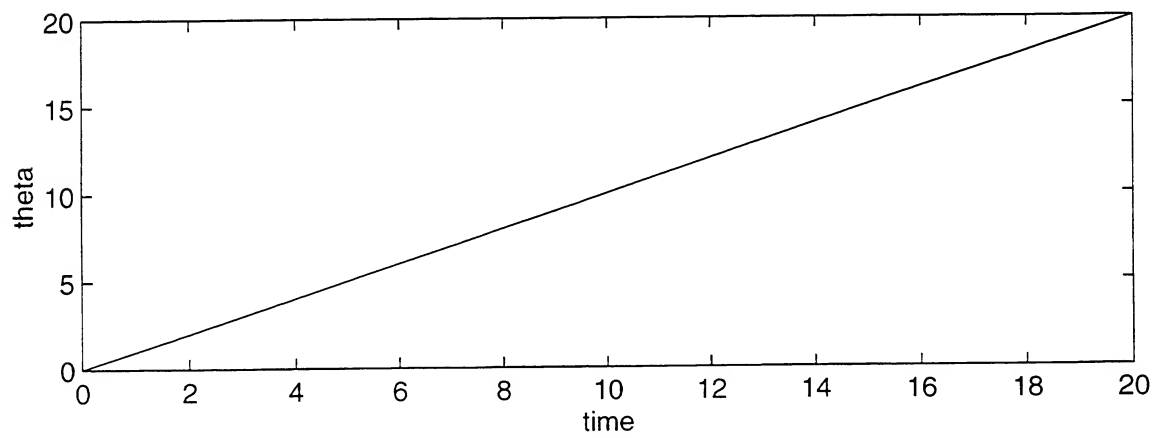
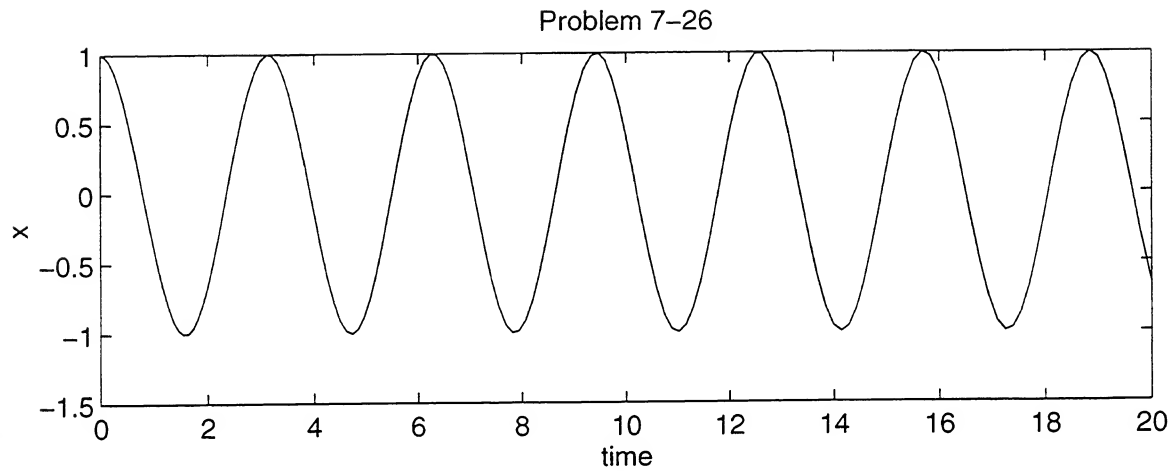
7-26 cont

```
clear
'Problem 7-26 Solution'
global m k
m=input('Enter the value of the mass m (kg) : ')
k=input('Enter the value of the spring constant k (N/m) : ')
x0=input('Enter the value of the four dimensional initial condition [x xd theta thetad] : ')
tf=input('Enter the value of the final time : ')
t0=0;
[t,x]=ode45('p726d',t0,tf,x0);
subplot(2,1,1);plot(t,x(:,1));xlabel('time');ylabel('x');title('Problem 7-26')
subplot(2,1,2);plot(t,x(:,3));xlabel('time');ylabel('theta');
```

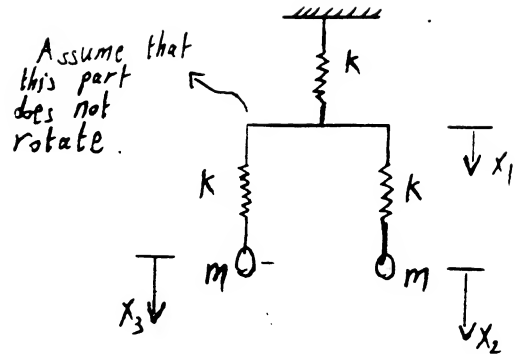
```
function xdot=p726d(t,x)
global m k
xdot=[x(2);
      -4*(k/m)*x(1);
      x(4);
      0];
```

7-26 cont.

$$m=k=1, \quad x_0 = [1 \ 0 \ 0 \ 1]$$



7-27



Kinetic energy $T = \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$

Potential energy $U = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_3 - x_1)^2 + \frac{1}{2} k (x_2 - x_1)^2$

Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) - \frac{\partial T}{\partial x_1} + \frac{\partial U}{\partial x_1} = 0 \Rightarrow kx_1 - k(x_3 - x_1) - k(x_2 - x_1) = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) - \frac{\partial T}{\partial x_2} + \frac{\partial U}{\partial x_2} = 0 \Rightarrow m\ddot{x}_2 + k(x_2 - x_1) = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_3} \right) - \frac{\partial T}{\partial x_3} + \frac{\partial U}{\partial x_3} = 0 \Rightarrow m\ddot{x}_3 + k(x_3 - x_1) = 0$$

In matrix form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since x_1 can be expressed in terms of x_2 and x_3 we write $(x_1 = \frac{1}{3}(x_2 + x_3))$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{k}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

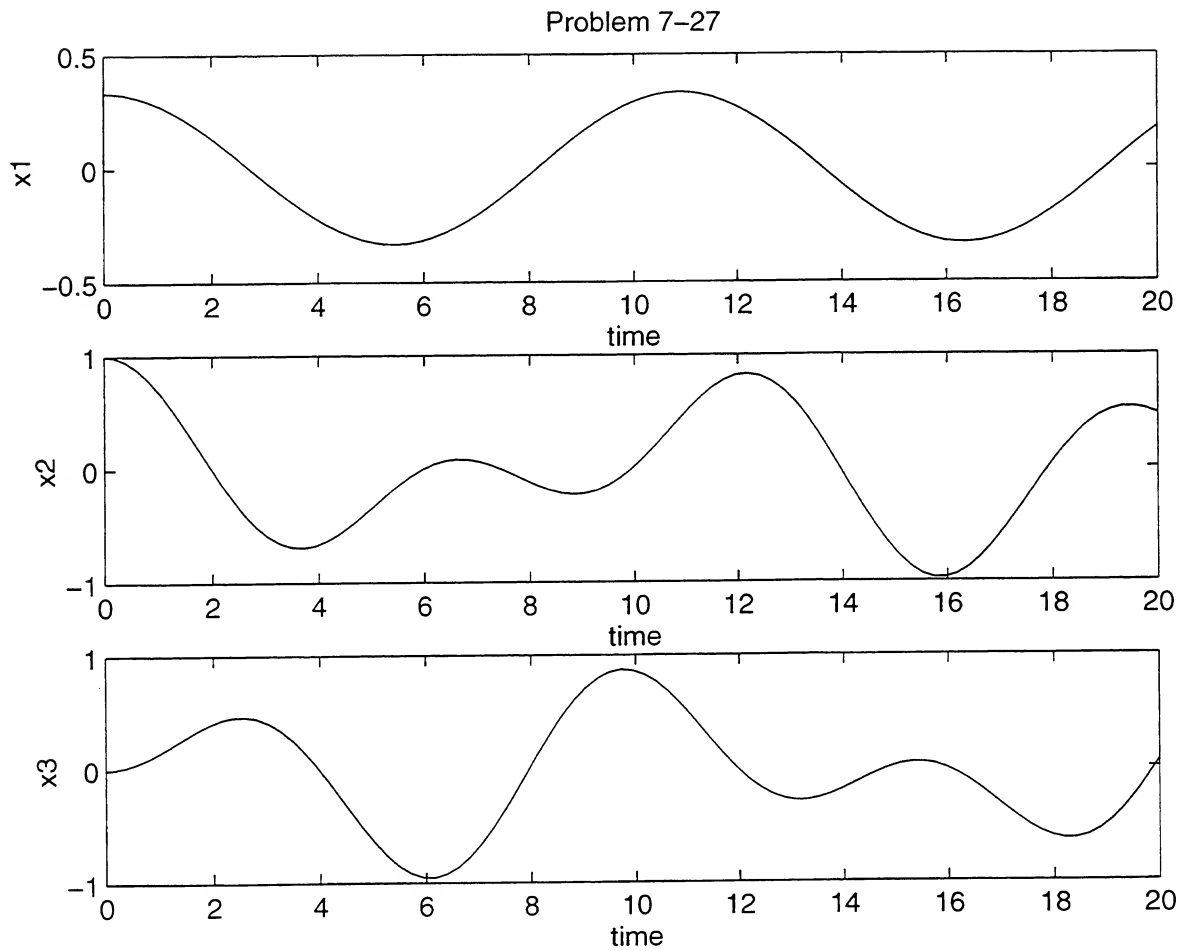
7-27 cont

```
clear
'Problem 7-27 Solution'
global m k
m=input('Enter the value of the mass m (kg) : ')
k=input('Enter the value of the spring constant k (N/m) : ')
x0=input('Enter the value of the four dimensional initial condition [x2 x2d x3 x3d] : ')
tf=input('Enter the value of the final time : ')
t0=0;
[t,x]=ode45('p727d',t0,tf,x0);
z=(1/3)*(x(:,1)+x(:,3));
subplot(3,1,1);plot(t,z);xlabel('time');ylabel('x1');title('Problem 7-27')
subplot(3,1,2);plot(t,x(:,1));xlabel('time');ylabel('x2');
subplot(3,1,3);plot(t,x(:,3));xlabel('time');ylabel('x3');
```

```
function xdot=p727d(t,x)
global m k
xdot=[x(2);
      -(k/(3*m))*(2*x(1)-x(3));
      x(4);
      -(k/(3*m))*(2*x(3)-x(1))];
```

7-27 cont

$$m=k=1, \quad x_0 = [1 \ 0 \ 0 \ 0]$$



8-1

Eq. in terms of stiffness

$$[-\lambda M + K]X = 0$$

when premultiplied by $K' = [a]$

Characteristic Eq. $[a][M] - \bar{\lambda} I]X = 0$ where $\bar{\lambda} = \frac{k}{\omega^2 m}$

$$\left| \begin{bmatrix} .5 & .5 & .5 \\ .5 & 1.5 & 1.5 \\ .5 & 1.5 & 2.5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \bar{\lambda} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (1-\bar{\lambda}) & .5 & .5 \\ 1 & (1.5-\bar{\lambda}) & 1.5 \\ 1 & 1.5 & (2.5-\bar{\lambda}) \end{vmatrix} = 0$$

Using first column as pivot

$$(1-\bar{\lambda}) \begin{vmatrix} (1.5-\bar{\lambda}) & 1.5 \\ 1.5 & (2.5-\bar{\lambda}) \end{vmatrix} - \begin{vmatrix} .5 & .5 \\ 1.5 & (2.5-\bar{\lambda}) \end{vmatrix} + \begin{vmatrix} .5 & .5 \\ (1.5-\bar{\lambda}) & 1.5 \end{vmatrix} = 0$$

$$(1-\bar{\lambda})(1.5-\bar{\lambda})(2.5-\bar{\lambda}) - 2.25(1-\bar{\lambda}) - (1.25 - .5\bar{\lambda}) + .75 + .75 - .75 + .5\bar{\lambda} = 0$$
$$-\bar{\lambda}^3 + 5\bar{\lambda}^2 - 4.50\bar{\lambda} + 1 = 0$$

8-2

Convert the matrix problem to the standard eigenvalue problem. In matlab

```
>>M=[ 2 0 0 ; 0 1 0 ; 0 0 1]
```

```
=
```

8-2 cont.

```
M =
```

```

2      0      0
0      1      0
0      0      1

```

```
>>K=[ 3 -1 0 ; -1 2 -1; 0 -1 1 ]
```

```
K =
```

```

3      -1      0
-1      2      -1
0      -1      1

```

```
>>t=inv(M)*K
```

```
t =
```

```

1.5000   -0.5000         0
-1.0000    2.0000   -1.0000
0   -1.0000    1.0000

```

```
>>[V,D]=eig (t)
```

```
V =
```

```

 $\phi_1$        $\phi_2$        $\phi_3$ 
0.7569    0.3031    0.2333
0.2189   -0.8422    0.5808
-0.6158    0.4458    0.7799

```

```
D =
```

```

1.3554         0         0
0    2.8892         0
0         0    0.2554

```

```
>>omega=sqrt(D) =  $\omega \sqrt{\frac{m}{K}}$ 
```

```
omega =
```

```

1.1642         0         0
0    1.6998         0
0         0    0.5053

```

These vectors can be normalized to give

$\phi_1 = \begin{pmatrix} -1.2291 \\ -1.3555 \\ 1.000 \end{pmatrix}$ $\phi_2 = \begin{pmatrix} 0.6799 \\ -1.8892 \\ 1.000 \end{pmatrix}$

$\phi_3 = \begin{pmatrix} 0.2991 \\ 0.7447 \\ 1.000 \end{pmatrix}$

8-3 Characteristic equation in terms of stiffness

$$\begin{bmatrix} (3-2\lambda_i) & -1 & 0 \\ -1 & (2-\lambda_i) & -1 \\ 0 & -1 & (1-\lambda_i) \end{bmatrix} = 0 \quad \lambda = \frac{\omega^2 m}{k}$$

For mode 2 substitute $\lambda_2 = 1.3554$

$$\begin{bmatrix} .2892 & -1 & 0 \\ -1 & .6446 & -1 \\ 0 & -1 & -.3554 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\div 1^{st}$ row by .2892 and add to 2^{nd} row

$$\begin{bmatrix} 1 & -3.4578 & 0 \\ 0 & -2.8132 & -1 \\ 0 & -1 & -.3554 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

From 3^{rd} row obtain $x_2 = -.3554$

" 1^{st} row $x_1 - .3554(-3.4578) = 0 \quad \therefore x_1 = -1.2289$

$$\therefore \phi_2 = \begin{Bmatrix} -1.2289 \\ -.3554 \\ 1.0000 \end{Bmatrix}$$

For ϕ_3 do same to obtain $\phi_3 = \begin{Bmatrix} .6800 \\ -1.889 \\ 1.0000 \end{Bmatrix}$

See Prob 8-2 computer results.

8-4

$$\begin{bmatrix} (3-2\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{bmatrix} = 0 \quad \text{For method of cofactors it is not necessary to alter the first row}$$

$$C_{11} = \begin{vmatrix} (2-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} \quad \text{let } \lambda_1 = .2554 \quad \text{then } C_{11} = .2991$$

$$C_{12} = - \begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} \quad C_{12} = .7447$$

8-4 Cont.

$$C_{13} = \begin{vmatrix} -1 & (2-\lambda) \\ 0 & -1 \end{vmatrix} = 1.0 \quad \therefore \phi_1 = \begin{Bmatrix} .2991 \\ .7447 \\ 1.00 \end{Bmatrix}$$

For ϕ_2 sub $\lambda_2 = 1.3554$ etc. into same cofactors
can also start with flexibility equation

8-5

Eg of motion in terms of cofactors of a column is
not proportional to the eigen vectors.

8-6

$$\left| -\left(\frac{\omega_m^2}{k}\right) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right| = \begin{vmatrix} (2-\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} = 0 \quad \lambda = \frac{\omega_m^2}{k}$$

Gaus Elim. $\lambda_1 = 0.198$

$$\begin{bmatrix} 1.802 & -1 & 0 \\ -1 & 1.199 & -1 \\ 0 & -1 & .802 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0 \quad \begin{aligned} x_2 &= .802 \\ x_1 &= \frac{.802}{1.802} = .445 \end{aligned}$$

$$\therefore \phi_1 = \begin{Bmatrix} .445 \\ .802 \\ 1.000 \end{Bmatrix}$$

When there are zeros in matrix
Gaus Elim. method is not necessary
for a 3x3 problem

$\lambda_2 = 1.555$

$$\begin{bmatrix} .445 & -1 & 0 \\ -1 & .445 & -1 \\ 0 & -1 & -.555 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0 \quad \begin{aligned} x_3 &= 1.0 \\ x_2 &= -.555 \\ x_1 &= -1.247 \end{aligned} \quad \phi_2 = \begin{Bmatrix} -1.247 \\ -.555 \\ 1.000 \end{Bmatrix}$$

$\lambda_3 = 3.247$

$$\begin{bmatrix} -1.247 & -1 & 0 \\ -1 & -1.247 & -1 \\ 0 & -1 & -2.247 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0 \quad \begin{aligned} x_3 &= 1.0 \\ x_2 &= -2.247 \\ x_1 &= 1.802 \end{aligned} \quad \phi_3 = \begin{Bmatrix} 1.802 \\ -2.247 \\ 1.000 \end{Bmatrix}$$

8-6 cont.

»D=[2 -1 0 ; -1 2 -1 ; 0 -1 1]

D =

2	-1	0
-1	2	-1
0	-1	1

»[V,D]=eig(D)

V =

ϕ_1	ϕ_2	ϕ_3
0.7370	-0.3280	0.5910
0.3280	-0.5910	-0.7370
-0.5910	-0.7370	0.3280

These can be normalized to get

$$\phi_1 = \begin{pmatrix} -1.2470 \\ -1.5548 \\ 1.000 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 1.4450 \\ 1.8019 \\ 1.000 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} -1.8018 \\ -2.247 \\ 1.000 \end{pmatrix}$$

D =

1.5550	0	0
0	0.1981	0
0	0	3.2470

»sqrt(D) = $\omega \sqrt{\frac{k}{m}}$

ans =

1.2470	0	0
0	0.4450	0
0	0	1.8019

»

8-7

From Example 6.1-1, the flexibility matrix is

$$\frac{1}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and the equation of motion is}$$

$$\left[\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} - \left(\frac{k}{\omega^2 m} \right) I \right] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \quad \text{Let } \bar{\lambda} = \frac{k}{\omega^2 m}$$

characteristic equation

$$\begin{vmatrix} (1-\bar{\lambda}) & 1 & 1 \\ 1 & (2-\bar{\lambda}) & 2 \\ 1 & 2 & (3-\bar{\lambda}) \end{vmatrix} = 0$$

$$\bar{\lambda}_1 = 5.0505$$

$$\bar{\lambda}_2 = 0.6431$$

$$\bar{\lambda}_3 = 0.3080$$

Subst. $\bar{\lambda} = 5.0505$

$$\begin{vmatrix} -4.0505 & 1 & 1 \\ 1 & -3.0505 & 2 \\ 1 & 2 & -2.0505 \end{vmatrix} \div 1^{\text{st}} \text{ row by } 4.0505 \quad \begin{vmatrix} -1 & .2469 & .2469 \\ 1 & -3.0505 & 2 \\ 1 & 2 & -2.0505 \end{vmatrix}$$

add rows 1 and 2 to form new row 2

$$\begin{vmatrix} -4.0505 & 1 & 1 \\ 0 & -2.8036 & 2.2469 \\ 1 & 2 & -2.0505 \end{vmatrix} \quad \text{next add rows 1+3 to form new row 3}$$

$$\begin{vmatrix} - & - & - \\ - & - & - \\ 0 & 2.2469 & -1.8036 \end{vmatrix}$$

result is

$$\begin{vmatrix} -4.0505 & 1 & 1 \\ 0 & -2.8036 & 2.2469 \\ 0 & 2.2469 & -1.8036 \end{vmatrix}$$

continue to reduce term 3,2 to zero

$$\begin{bmatrix} -4.0505 & 1 & 1 \\ 0 & -2.8036 & 2.2469 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ 1.0 \end{Bmatrix} = 0$$

$$x_2 = \frac{2.2469}{2.8036} = .8014$$

$$x_1 = \frac{1.8014}{4.0505} = .4447$$

$$\phi_1 = \begin{Bmatrix} .4447 \\ .8014 \\ 1.000 \end{Bmatrix} \quad \text{see Prob 8-6 for } \phi_2 \neq \phi_3$$

$$\text{repeat with } \bar{\lambda}_2 = 0.6431 \quad \text{and } \bar{\lambda}_3 = .3080 \quad \phi_2 = \begin{Bmatrix} -1.247 \\ -.5548 \\ 1.000 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 1.818 \\ -2.247 \\ 1.000 \end{Bmatrix}$$

8-8 Change k_i and m_i as per assigned by instructor

8-9 With $k_i = k$ and $m_i = m$, the equation of motion is

$$\left\{ \lambda \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \right\} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \lambda = \frac{\omega^2 m}{k}$$

Characteristic equation

$$\begin{vmatrix} (3-\lambda) & -1 & -1 & 0 \\ -1 & (2-\lambda) & -1 & 0 \\ -1 & -1 & (3-\lambda) & -1 \\ 0 & 0 & -1 & (1-\lambda) \end{vmatrix} = (3-\lambda) \begin{vmatrix} (2-\lambda) & -1 & 0 \\ -1 & (3-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ -1 & (3-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} - \begin{vmatrix} -1 & (2-\lambda) & 0 \\ -1 & -1 & -1 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = 0$$

$$= (3-\lambda)(2-\lambda) \begin{vmatrix} (3-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} + (3-\lambda) \begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} - \begin{vmatrix} (3-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} + (2-\lambda) \begin{vmatrix} -1 & -1 \\ 0 & (1-\lambda) \end{vmatrix} = 0$$

$$= [(3-\lambda)(2-\lambda)-1][(3-\lambda)(1-\lambda)-1] - [(7-2\lambda)(1-\lambda)] = 0$$

$$\therefore \lambda^4 - 9\lambda^3 + 25\lambda^2 - 21\lambda + 3 = 0$$

The eigenvalues can either be determined from the roots of the Characteristic equation or directly using the command eig.

>>c=[1 -9 25 -21 3]

c =

1 -9 25 -21 3

>>roots(c)

ans =

4.1796
3.4882
1.1538
0.1783

8-9 cont.

215

```
»D=[ 3 -1 -1 0 ; -1 2 -1 0 ; -1 -1 3 -1 ; 0 0 -1 1]
```

D =

3	-1	-1	0
-1	2	-1	0
-1	-1	3	-1
0	0	-1	1

```
»[V,D]=eig(D)
```

V =

0.6371	-0.5800	0.3679	-0.3497
-0.6679	-0.0883	0.5672	-0.4737
0.3569	0.7725	0.1120	-0.5131
-0.1434	-0.2430	-0.7283	-0.6245

D =

3.4882	0	0	0
0	4.1796	0	0
0	0	1.1538	0
0	0	0	0.1783

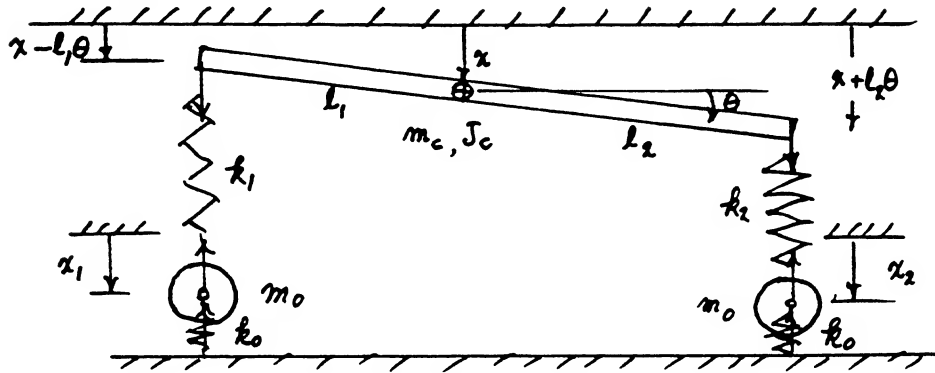
»

8-10

Optional (See computer programs for Prob. 8-9)

8-11

4 DOF Auto



$$m_c \ddot{x}_c = k_1 [x_1 - (x - l_1 \theta)] + k_2 [x_2 - (x + l_2 \theta)]$$

$$J_c \ddot{\theta} = -k_1 l_1 [x_1 - (x - l_1 \theta)] + k_2 l_2 [x_2 - (x + l_2 \theta)]$$

$$m_0 \ddot{x}_1 = -k_0 x_1 - k_1 [x_1 - (x - l_1 \theta)]$$

$$m_0 \ddot{x}_2 = -k_0 x_2 - k_2 [x_2 - (x + l_2 \theta)]$$

$$\begin{bmatrix} m_c \\ J_c \\ m_0 \\ m_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 l_2 - k_1 l_1) & -k_1 & -k_2 \\ (k_2 l_2 - k_1 l_1) & (k_1 l_1^2 + k_2 l_2^2) & k_1 l_1 & -k_2 l_2 \\ -k_1 & k_1 l_1 & (k_1 + k_0) & 0 \\ -k_2 & -k_2 l_2 & 0 & (k_2 + k_0) \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

8-12

4 DOF Auto

$$m = \begin{bmatrix} 100 & | & \\ \hline 1600 & | & \\ \hline & | & 4.969 \\ & | & \\ & | & 4.969 \end{bmatrix}$$

$$k = \begin{bmatrix} 5000 & 3500 & | & -2400 & -2600 \\ 3500 & 127250 & | & 10800 & -14300 \\ \hline -2400 & 10800 & | & 40800 & 0 \\ -2600 & -14300 & | & 0 & 41000 \end{bmatrix}$$

8-12 cont.

=

```
>>M=[ 100 0 0 0 ; 0 1600 0 0 ; 0 0 4.969 0 ; 0 0 0 4.969]
```

M =

1.0e+03 *

0.1000	0	0	0
0	1.6000	0	0
0	0	0.0050	0
0	0	0	0.0050

```
>>K=[ 5000 3500 -2400 -2600; 3500 127250 10800 -14300;
-2400 10800 40800 0 ; -2600 -14300 0 41000]
```

K =

5000	3500	-2400	-2600
3500	127250	10800	-14300
-2400	10800	40800	0
-2600	-14300	0	41000

```
>>D=inv(M)*K
```

D =

1.0e+03 *

0.0500	0.0350	-0.0240	-0.0260
0.0022	0.0795	0.0067	-0.0089
-0.4830	2.1735	8.2109	0
-0.5232	-2.8778	0	8.2512

```
>>[v,d]=eig(D)
```

v =

0.9940	-0.0031	0.0030	-0.6996
-0.0671	-0.0011	-0.0008	-0.6473
0.0767	-0.0202	-0.9998	0.1314
0.0398	0.9998	-0.0203	-0.2727

rotation wheel vibration translation

8-12 cont.

$$d = \begin{array}{ccccc} & \text{rotation} & \text{wheel vibration} & & \text{translation} \\ & 0.0447 & \begin{array}{cc} 0 & 0 \end{array} & & 0 \\ & 0 & 8.2560 & 0 & 0 \\ & 0 & 0 & 8.2141 & 0 \\ & 0 & 0 & 0 & 0.0768 \end{array}$$

8-13

$$m = \begin{bmatrix} 500 & 0 \\ 0 & 100 \end{bmatrix} \quad [a] = \frac{l^3}{48EI} \begin{bmatrix} 1 & -1.5 \\ -1.5 & 6 \end{bmatrix}$$

$$[a]^{-1} = [k] = \frac{48EI}{l^3} \cdot \frac{1}{3.75} \begin{bmatrix} 6 & 1.5 \\ 1.5 & 1 \end{bmatrix} = \frac{12.80EI}{l^3} \begin{bmatrix} 6 & 1.5 \\ 1.5 & 1 \end{bmatrix}$$

$$\text{Let } \bar{\lambda} = \left(\frac{12.80EI}{\omega^2 l^3} \right) \quad \text{and}$$

$$\left| \begin{bmatrix} 500 & 0 \\ 0 & 100 \end{bmatrix} - \bar{\lambda} \begin{bmatrix} 6 & 1.5 \\ 1.5 & 1 \end{bmatrix} \right| = 0$$

$$\gg M = \begin{bmatrix} 500 & 0 \\ 0 & 100 \end{bmatrix}$$

$$M =$$

$$\begin{array}{cc} 500 & 0 \\ 0 & 100 \end{array}$$

$$\gg K = \begin{bmatrix} 6 & 1.5 \\ 1.5 & 1 \end{bmatrix}$$

$$K =$$

$$\begin{array}{cc} 6.0000 & 1.5000 \\ 1.5000 & 1.0000 \end{array}$$

$$\gg d = \text{inv}(K) * M$$

$$d =$$

$$\begin{array}{cc} 133.3333 & -40.0000 \\ -200.0000 & 160.0000 \end{array}$$

8-13 Cont

$$\bar{\lambda} = \frac{12.8 EI}{\omega^2 L^3}$$

»[v,d]=eig(d)

v =

$$\begin{bmatrix} -0.4605 & 0.3597 \\ -0.8876 & -0.9331 \end{bmatrix}$$

d =

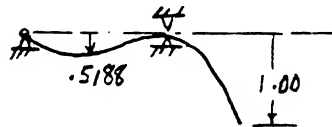
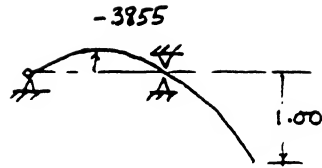
$$\begin{bmatrix} 56.2356 & 0 \\ 0 & 237.0977 \end{bmatrix}$$

$$\omega_1 = .2323 \sqrt{\frac{EI}{L^3}}$$

$$\phi_1 = \begin{Bmatrix} -.3855 \\ 1.000 \end{Bmatrix}$$

$$\omega_2 = .4771 \sqrt{\frac{EI}{L^3}}$$

$$\phi_2 = \begin{Bmatrix} .5188 \\ 1.000 \end{Bmatrix}$$



8-14

Equation of motion is

$$m \begin{bmatrix} 3 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 6 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

»M=[3 0 0 ; 0 2 0 ; 0 0 1]

M =

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

»K=[6 -2 0; -2 3 -1; 0 -1 1]

K =

$$\begin{bmatrix} 6 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

8-14 cont

$$\gg d = \text{inv}(K) * M$$

d =

$$\begin{bmatrix} 0.7500 & 0.5000 & 0.2500 \\ 0.7500 & 1.5000 & 0.7500 \\ 0.7500 & 1.5000 & 1.7500 \end{bmatrix}$$

$$\gg [v, d] = \text{eig}(d) \quad \bar{\lambda} = \left(\frac{K}{\omega^2 m} \right)$$

v =

$$\begin{bmatrix} 0.2160 & -0.6256 & 0.4337 \\ 0.5376 & 0.6743 & 0.3636 \\ 0.8150 & -0.3924 & -0.8245 \end{bmatrix}$$

d =

$$\begin{bmatrix} 2.9382 & 0 & 0 \\ 0 & 0.3678 & 0 \\ 0 & 0 & 0.6939 \end{bmatrix} \quad \begin{aligned} \omega_1 &= .5834 \sqrt{K/m} \\ \omega_2 &= 1.649 \sqrt{K/m} \\ \omega_3 &= 1.201 \sqrt{K/m} \end{aligned}$$

8-15

The flexibility matrix is given in Example 6.1-3

$$[a] = \frac{L^3}{3EI} \begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix} \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \leftarrow \text{free end.}$$

Inverse of above matrix is

$$\frac{L^3}{3EI} \begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix}^{-1} = \frac{3EI}{L^3} \begin{bmatrix} .5385 & -1.231 & .9231 \\ & 3.385 & -3.539 \\ & & 6.154 \end{bmatrix}$$

8-15 cont

$$\gg M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

M =

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\gg K = \begin{bmatrix} .5385 & -1.231 & .9231 \\ -1.231 & 3.385 & -3.539 \\ .9231 & -3.539 & 6.154 \end{bmatrix}$$

K =

$$\begin{bmatrix} 0.5385 & -1.2310 & 0.9231 \\ -1.2310 & 3.3850 & -3.5390 \\ 0.9231 & -3.5390 & 6.1540 \end{bmatrix}$$

$$\gg d = \text{inv}(K) * M$$

$$\lambda = \frac{3EI}{\omega^2 l^3}$$

d =

$$\begin{bmatrix} 27.1245 & 14.0695 & 4.0223 \\ 14.0695 & 8.0387 & 2.5124 \\ 4.0223 & 2.5124 & 1.0040 \end{bmatrix}$$

$$\gg [v, d] = \text{eig}(d)$$

v =

$$\begin{array}{ccc} \phi_1 & \phi_2 & \phi_3 \\ 0.4526 & 0.1737 & 0.8746 \\ -0.6822 & -0.5642 & 0.4651 \\ -0.5743 & 0.8071 & 0.1369 \end{array}$$

$$\phi_1 = \begin{pmatrix} 1.00 \\ -1.504 \\ -1.269 \end{pmatrix} \leftarrow \text{free end}$$

$$\phi_2 = \begin{pmatrix} 1.00 \\ -3.248 \\ 4.647 \end{pmatrix} \quad \phi_3 = \begin{pmatrix} 1.000 \\ .5316 \\ .1564 \end{pmatrix}$$

d =

$$\begin{bmatrix} 0.8180 & 0 & 0 \\ 0 & 0.1133 & 0 \\ 0 & 0 & 35.2359 \end{bmatrix}$$

$$\gg \omega_1 = 1.916 \sqrt{\frac{EI}{l^3}} \quad \omega_3 = .2925 \sqrt{\frac{EI}{l^3}} \\ \omega_2 = 5.146 \sqrt{\frac{EI}{l^3}}$$

8-20

$$3m\ddot{x}_1 = -4kx_1 + k(x_2 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

$$M = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} 2.2361 & -.4472 \\ 0 & 1.3416 \end{bmatrix} \quad \text{see App C5}$$

$$Q^{-1} = \begin{bmatrix} .44721 & .14907 \\ 0 & .74535 \end{bmatrix} \quad [Q^{-T} M Q^{-1} - \bar{\lambda} I] Y = 0 \quad \bar{\lambda} = \frac{k}{\omega^2 m}$$

$$Q^{-T} M Q^{-1} = \begin{bmatrix} .4472 & 0 \\ .1491 & .7454 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .4472 & .1491 \\ 0 & .7454 \end{bmatrix} = \begin{bmatrix} .600 & .200 \\ .200 & .6221 \end{bmatrix} = \tilde{A}$$

standard form $[\tilde{A} - \bar{\lambda} I] Y = 0 \quad Y = QX \quad \text{or} \quad X = Q^{-1}Y$

8-21

$$K = k \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad Q = U = \begin{bmatrix} 1.732 & -.5774 & -.5774 \\ 0 & 1.2910 & -1.0328 \\ 0 & 0 & .7746 \end{bmatrix}$$

see App. C5

$$Q^{-1} = U^{-1} = \begin{bmatrix} .5773 & .2582 & .7746 \\ 0 & .7746 & 1.0328 \\ 0 & 0 & 1.2910 \end{bmatrix}$$

$\gg k = [3 \ -1 \ -1 ; -1 \ 2 \ -1 ; -1 \ -1 \ 2]$

k =

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$\gg q = \text{chol}(k)$

q =

$$\begin{bmatrix} 1.7321 & -0.5774 & -0.5774 \\ 0 & 1.2910 & -1.0328 \\ 0 & 0 & 0.7746 \end{bmatrix}$$

8-16

$$M = J \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$K = K \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\gg k = [1 \ -1 \ 0 ; -1 \ 2 \ -1; 0 \ -1 \ 2]$$

$$k =$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\bar{\lambda} = \frac{K}{\omega^2 J}$$

$$\omega_i = \frac{1}{\sqrt{\bar{\lambda}}} \sqrt{\frac{K}{J}}$$

$$\gg d = \text{inv}(k)$$

$$d =$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\gg [v, d] = \text{eig}(d)$$

$$v =$$

$$\begin{array}{ccc} \phi_1 & \phi_2 & \phi_3 \\ 0.3280 & -0.5910 & 0.7370 \\ -0.7370 & 0.3280 & 0.5910 \\ 0.5910 & 0.7370 & 0.3280 \end{array}$$

$$\phi_1 = \begin{pmatrix} 1.000 \\ -2.247 \\ 1.802 \end{pmatrix} \leftarrow \text{Free end}$$

$$\phi_2 = \begin{pmatrix} 1.000 \\ -5.555 \\ -1.247 \end{pmatrix}$$

$$\phi_3 = \begin{pmatrix} 1.000 \\ 0.8019 \\ 0.4450 \end{pmatrix}$$

$$d =$$

$$\begin{array}{ccc} \omega_1 & \omega_2 & \omega_3 \\ 0.3080 & 0 & 0 \\ 0 & 0.6431 & 0 \\ 0 & 0 & 5.0489 \end{array}$$

$$\omega_1 = 1.802 \sqrt{\frac{K}{J}} \quad \omega_2 = 1.247 \sqrt{\frac{K}{J}}$$

$$\omega_3 = 0.445 \sqrt{\frac{K}{J}}$$

8-17

Equation of motion

$$-\omega^2 m \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} + \frac{I}{L} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

8-17 cont

=

»K=[2 -1 0 0 ; -1 2 -1 0 ; 0 -1 2 -1; 0 0 -1 2]

K =

$$\bar{\lambda} = \frac{T}{\omega^2 m l}$$

2	-1	0	0
-1	2	-1	0
0	-1	2	-1
0	0	-1	2

»d=inv(K)

d =

0.8000	0.6000	0.4000	0.2000
0.6000	1.2000	0.8000	0.4000
0.4000	0.8000	1.2000	0.6000
0.2000	0.4000	0.6000	0.8000

»[v,d]=eig(d)

v =

0.3717	0.6015	0.6015	-0.3717
0.6015	0.3717	-0.3717	0.6015
0.6015	-0.3717	-0.3717	-0.6015
0.3717	-0.6015	0.6015	0.3717

d =

2.6180	0	0	0	$\omega_1 = 0.618$
0	0.7236	0	0	$\omega_2 = 1.175 \sqrt{\frac{T}{me}}$
0	0	0.3820	0	$\omega_3 = 1.617 \sqrt{\frac{T}{me}}$
0	0	0	0.2764	$\omega_4 = 1.902$

8-18

```
>>x=[2 -1 ; -1 4]
```

x =

```
     2     -1
    -1      4
```

```
>>U=chol(x)
```

U =

```
    1.4142    -0.7071
         0     1.8708
```

```
>>U'*U      check calculation
```

ans =

```
    2.0000    -1.0000
   -1.0000     4.0000
```

8-19

```
>>x=[3 -1 ; -1 1]
```

x =

```
     3     -1
    -1      1
```

```
>>U=chol(x)
```

U =

```
    1.7321    -0.5774
         0     0.8165
```

8-21 Cont

»inv(q)

ans =

```
0.5774    0.2582    0.7746
         0    0.7746    1.0328
         0         0    1.2910
```

8-22

$$m \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{Decompose M}$$

$$\text{Let } \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = Q^T Q \quad Q = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1.7321 & 0 \\ 0 & 1.4142 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} .5773 & 0 \\ 0 & .7072 \end{bmatrix} = Q^{-T} \quad \left[-\left(\frac{\omega^2 m}{k}\right) I + Q^T K Q^{-1} \right] Y = 0$$

$$Q^{-T} K Q^{-1} = \begin{bmatrix} .5773 & 0 \\ 0 & .707 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} .5773 & 0 \\ 0 & .707 \end{bmatrix} = \begin{bmatrix} 1.00 & -.4082 \\ -.4082 & 1.00 \end{bmatrix} = \tilde{A}$$

»m=[3 0 ; 0 2]

m =

```
3    0
0    2
```

»k=[3 -1 ; -1 2]

k =

```
3    -1
-1   2
```

»q=sqrt(m)

q =

```
1.7321    0
0    1.4142
```

8-22 cont

$\gg a = \text{inv}(q) * k * \text{inv}(q)$

$a =$

1.0000 -0.4082
-0.4082 1.0000

$\gg [v, d] = \text{eig}(a)$

$$\lambda = \frac{\omega^2 m}{K}$$

$v =$

ϕ_2 ϕ_1
-0.7071 -0.7071
0.7071 -0.7071

$$\phi_2 = \begin{pmatrix} -0.817 \\ 1.00 \end{pmatrix} \quad \phi_1 = \begin{pmatrix} 0.817 \\ 1.00 \end{pmatrix}$$

$d =$

1.4082 0 $\omega_2 = 1.187 \sqrt{K/m}$
0 0.5918 $\omega_1 = 0.7693 \sqrt{K/m}$

8-23

$$\left[-\frac{\omega^2 m}{K} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \text{Decompose}_K$$

see example 8.8-2

$\gg m = [4 \ 0 \ 0; 0 \ 2 \ 0; 0 \ 0 \ 1]$

$m =$

4 0 0
0 2 0
0 0 1

$\gg k = [4 \ -1 \ 0; -1 \ 2 \ -1; 0 \ -1 \ 1]$

$k =$

4 -1 0
-1 2 -1
0 -1 1

8-23 w/m
 »a=chol(k)

a =

2.0000	-0.5000	0
0	1.3229	-0.7559
0	0	0.6547

»ain=inv(a)

ain =

0.5000	0.1890	0.2182
0	0.7559	0.8729
0	0	1.5275

»atild=ain'*m*ain

atild =

1.0000	0.3780	0.4364
0.3780	1.2857	1.4846
0.4364	1.4846	4.0476

»[v,d]=eig(atild)

v =

0.4232	-0.8944	0.1447
-0.8516	-0.3381	0.4006
0.3094	0.2928	0.9047

d =

0.5585	0	0
0	1.0000	0
0	0	4.7749

$$\bar{\lambda} = \frac{k}{\omega^2 m}$$

$$\omega = \sqrt{\frac{1}{\bar{\lambda}}} \sqrt{\frac{k}{m}}$$

$$\omega_1 = 1.3381 \sqrt{k/m}$$

$$\omega_2 = 1.00 \sqrt{k/m}$$

$$\omega_3 = 0.4576 \sqrt{k/m}$$

8-23 cont.

```
>>x=qin*v
```

x =

```
0.1182 -0.4472 0.3455
-0.3737 -0.0000 1.0926
0.4726 0.4472 1.3820
```

$$x = Q^{-1}y$$

where

$$[Q^T m Q^{-1} - \bar{\lambda} I] y = 0$$

8-24

```
>>m=[4 1 0; 1 4 1; 0 1 2]
```

m =

```
4 1 0
1 4 1
0 1 2
```

The matrix problem
is

$$[-\lambda I + Q^T K Q^{-1}] y = 0$$

$$\lambda = \frac{\omega^2 m}{K}$$

$$x = Q^{-1}y$$

```
>>k=[2 -1 0 ; -1 2 -1 ; 0 -1 1]
```

k =

```
2 -1 0
-1 2 -1
0 -1 1
```

```
>>q=chol(m)
```

q =

```
2.0000 0.5000 0
0 1.9365 0.5164
0 0 1.3166
```

8-24 cont

```
>qin=inv(q)
```

```
qin =
```

```
    0.5000   -0.1291    0.0506  
         0    0.5164   -0.2025  
         0         0    0.7596
```

```
>a=qin'*k*qin
```

```
a =
```

dynamic matrix

```
    0.5000   -0.3873    0.1519  
   -0.3873    0.7000   -0.6668  
    0.1519   -0.6668    0.9923
```

```
>[v,d]=eig(a)
```

```
v =
```

```
    0.4887    0.8165    0.3074  
    0.7480   -0.2108   -0.6293  
    0.4490   -0.5375    0.7138
```

```
d =
```

```
    0.0467         0         0  
         0    0.5000         0  
         0         0    1.6456
```

```
>x=qin*v
```

```
x =
```

$x = Q^{-1}y$

```
    0.1705    0.4082    0.2711  
    0.2953   -0.0000   -0.4695  
    0.3410   -0.4082    0.5422
```

```
>>
```

8-25

$$[Q^T m Q - \bar{\lambda} I] \gamma = 0 \quad \bar{\lambda} = \frac{k}{\omega^2 m} \quad x = Q^T \gamma$$

```
>>k=[2 -1 0 ; -1 2 -1 ; 0 -1 1 ]
```

k =

```

2     -1     0
-1     2     -1
0     -1     1
```

```
>>m=[4 1 0 ; 1 4 1 ; 0 1 2]
```

m =

```

4     1     0
1     4     1
0     1     2
```

```
>>q=chol(k)
```

q =

```

1.4142   -0.7071     0
0         1.2247   -0.8165
0         0         0.5774
```

```
>>qin=inv(q)
```

qin =

```

0.7071    0.4082    0.5774
0         0.8165    1.1547
0         0         1.7321
```

```
>>a=qin'*m*qin
```

a =

```

2.0000    1.7321    2.4495
1.7321    4.0000    7.0711
2.4495    7.0711   18.0000
```

8-25 cont

$\gg [v,d] = \text{eig}(a)$

$v =$

```
-0.8165    -0.1494    0.5577
-0.4714    -0.3851   -0.7934
 0.3333    -0.9107    0.2440
```

$d =$

```
2.0000      0      0
      0 21.3923      0
      0      0 0.6077
```

$$\bar{\omega}_i = \frac{1}{\sqrt{\lambda}} \sqrt{\frac{k}{m}}$$

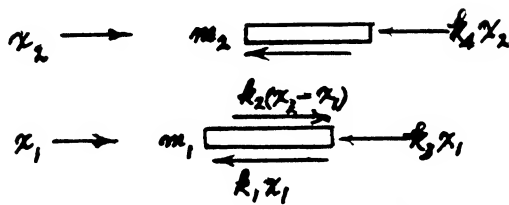
$\gg x = qin * v$

$x =$

```
-0.5774    -0.7887    0.2113
 0.0000    -1.3660   -0.3660
 0.5774    -1.5774    0.4226
```

$$x = Q^{-1} y$$

8-26



$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_4 x_2$$

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) - k_3 x_1$$

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2 + k_3) & -k_2 \\ -k_2 & (k_2 + k_4) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

8-26 Cont

```
>>m=[ 1 0 ; 0 1]
```

m =

```
    1    0
    0    1
```

```
>>k=[ 3 -1 ; -1 2 ]
```

k =

```
    3    -1
   -1     2
```

```
>>[v,d]=eig(k)
```

v =

```
  -0.8507  -0.5257
   0.5257  -0.8507
```

d =

```
  3.6180    0
    0    1.3820
```

$$\lambda = \frac{\omega^2 m}{\kappa}$$

$$\frac{8-21}{}$$

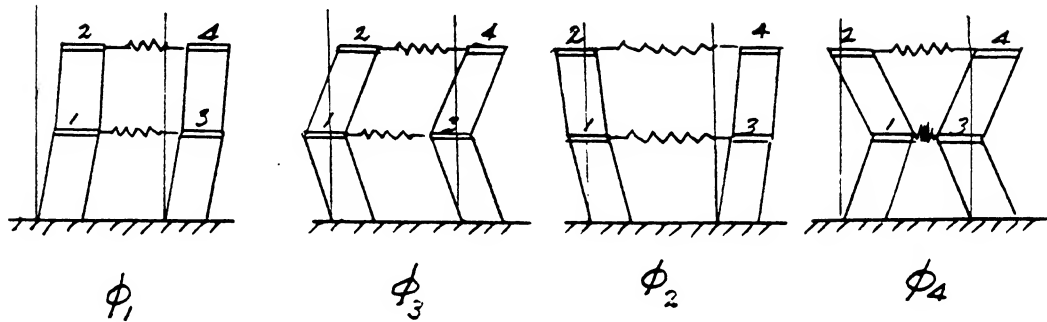
$$M = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} m. \quad K = k \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) + k(x_3 - x_1)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_4 - x_2)$$

$$m\ddot{x}_3 = -kx_3 + k(x_4 - x_3) - k(x_3 - x_1)$$

$$m\ddot{x}_4 = -k(x_4 - x_3) - k(x_4 - x_2)$$



$$\gg m = \text{eye}(4)$$

$$m =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gg k = [3 \ -1 \ -1 \ 0 ; -1 \ 2 \ 0 \ -1 ; -1 \ 0 \ 3 \ -1 ; 0 \ -1 \ -1 \ 2]$$

$$k =$$

$$\begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$\sigma - \alpha + \cos t$

\equiv

$\gg [v, d] = \text{eig}(k)$

$v =$

ϕ_4	ϕ_2	ϕ_3	ϕ_1
0.6015	-0.3717	+0.6015	+0.3717
-0.3717	-0.6015	-0.3717	+0.6015
-0.6015	0.3717	-0.6015	+0.3717
+0.3717	0.6015	+0.3717	+0.6015

$d =$

2.6180	0	0	0
0	2.3820	0	0
0	0	4.6180	0
0	0	0	0.3820

$$\underline{9-1} \quad c = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{444}{.372}} = 34.55 \text{ m/s}$$

9-2

$$f_n = \frac{n}{2l} \sqrt{\frac{T}{\rho}}, \quad n = 1, 2, 3, \dots$$

9-3

$$\text{gen sol. } y(x, t) = (A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}) \sin \omega t$$

$$\text{At } x=0 \quad y=0 \quad \therefore B=0$$

$$\text{At } x=l \quad y=y(l, t) \text{ of spring mass}$$

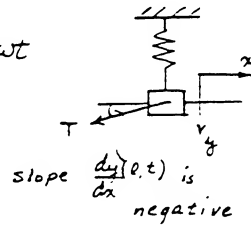
$$\text{Vertical force} = -T \frac{dy}{dx}(l, t)$$

$$= (-TA \frac{\omega}{c} \cos \frac{\omega l}{c}) \sin \omega t$$

$$= m \ddot{y} + ky$$

$$\therefore y(l) = -\frac{TA \frac{\omega}{c} \cos \frac{\omega l}{c}}{k - m\omega^2} = -\frac{TA \frac{\omega}{c} \cos \frac{\omega l}{c}}{m\omega^2 (1 - \frac{\omega^2}{\omega_n^2})}, \quad \omega_n = \sqrt{\frac{k}{m}}$$

$$\text{or } \tan \frac{\omega l}{c} = -\left(\frac{T}{kl}\right) \left[\frac{(\frac{\omega l}{c})}{1 - (\frac{\omega l}{c})^2 (\frac{mc^2}{kl^2})} \right]$$



9-4

$$y_1 = a \cos kx \sin \omega t, \quad y_2 = a \cos\left(\frac{k}{2} + \frac{\omega}{2}\right) \sin\left(\omega t + \frac{T}{2}\right) = -a \sin kx \cos \omega t$$

$$y = y_1 + y_2 = a [\cos kx \sin \omega t - \sin kx \cos \omega t]$$

$$= a \sin(\omega t - kx) = a \sin k \left(\frac{\omega}{k} t - x \right) \therefore c = \frac{\omega}{k}$$

9-5

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{200 \times 10^9}{7810}} = 5060 \text{ m/s} = 16,600 \text{ ft/s}$$

9-6

$$dT \approx \rho g dx$$

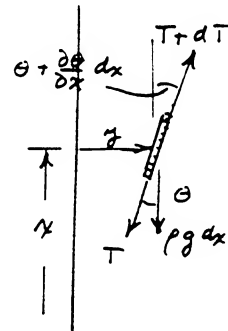
$$(T + dT)(\theta + \frac{\partial \theta}{\partial x} dx) - T\theta = \rho dx \ddot{y}$$

$$\therefore T \frac{\partial \theta}{\partial x} + \rho g \theta = \rho \ddot{y}$$

$$\theta = \frac{\partial y}{\partial x}, \quad T = \rho g x$$

$$\rho g x \frac{\partial^2 y}{\partial x^2} + \rho g \frac{\partial y}{\partial x} = \rho \ddot{y}$$

$$\ddot{y} = g \left(x \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} \right)$$



9-7

$$y = Y(x) \cos \omega t$$

$$\ddot{y} = -\omega^2 Y(x) \cos \omega t$$

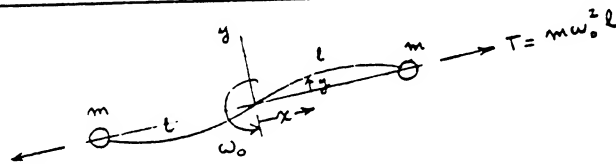
$$-\omega^2 Y(x) = g \left(x \frac{d^2 Y(x)}{dx^2} + \frac{dY(x)}{dx} \right)$$

$$\text{Let } z^2 = \frac{4\omega^2}{g} x \quad dx = \frac{g}{4\omega^2} 2z dz, \quad (dx)^2 = \left(\frac{g 2z}{4\omega^2} \right)^2 dz^2$$

$$-\omega^2 Y = g \left\{ \frac{g}{4\omega^2} z^2 \frac{d^2 Y}{\frac{g 2z}{4\omega^2} dz^2} + \frac{2\omega^2}{g z} \frac{dY}{dz} \right\}$$

$$\therefore \frac{d^2 Y}{dz^2} + \frac{1}{z} \frac{dY}{dz} + Y = 0 \quad \text{Bessel's D.E.}$$

9-8



Assume mode shape as shown. Accel. at y is $\ddot{y} - y\omega_0^2$ in lateral direction

$$T \frac{d^2 y}{dx^2} = \rho (\ddot{y} - y\omega_0^2) \quad \text{Let } y = Y(x) e^{i\omega t}$$

$$\frac{d^2 Y}{dx^2} + \left[\left(\frac{\omega}{c} \right)^2 + \left(\frac{\omega_0}{c} \right)^2 \right] Y(x) = 0 \quad c = \sqrt{\frac{T}{\rho}}$$

$$Y(0) = 0 \quad \therefore B = 0 \quad \text{and } Y(x) = A \sin \Omega x + 0$$

$$\text{where } \Omega = \sqrt{\left(\frac{\omega}{c} \right)^2 + \left(\frac{\omega_0}{c} \right)^2}$$

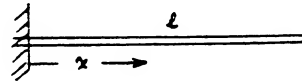
$$Y(l) = 0 \quad \therefore \sin \Omega l = 0$$

$$\sqrt{\left(\frac{\omega}{c} \right)^2 + \left(\frac{\omega_0}{c} \right)^2} l = \pi \quad \omega^2 + \left(\frac{\pi c}{l} \right)^2 = \omega_0^2$$

$$\therefore \underline{\underline{\omega^2 = \left(\frac{\pi}{l} \right)^2 \left(\frac{m \omega_0^2 l}{\rho} \right) - \omega_0^2}}$$

9-9

$$u = \left(A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c} \right) \sin \omega t$$



$$u(0) = 0 \quad \therefore B = 0$$

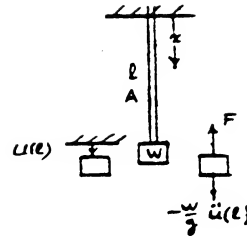
$$\sigma = E \left(\frac{du}{dx} \right)_{x=l} = 0 \quad \therefore \frac{\omega}{c} \cos \frac{\omega l}{c} = 0$$

$$\frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots = \left(n + \frac{1}{2} \right) \pi \quad n = 0, 1, 2, \dots$$

$$\underline{\underline{f_n = \frac{\omega}{2\pi} = \left(n + \frac{1}{2} \right) \frac{c}{2l}, \quad n = 0, 1, 2, \dots}}$$

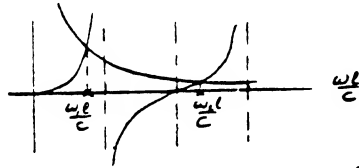
9-10

$$\begin{aligned}
 u &= C \sin \frac{\omega x}{c} \sin \omega t \\
 F &= A\sigma = AE \frac{\partial u}{\partial x} = AEC \frac{\omega}{c} \cos \frac{\omega x}{c} \sin \omega t \\
 &= -\frac{W}{g} \ddot{u}(l) \\
 \therefore AE \frac{\omega}{c} \cos \frac{\omega l}{c} &= \omega^2 \frac{W}{g} \sin \frac{\omega l}{c} \\
 \frac{\omega l}{c} \tan \frac{\omega l}{c} &= \frac{Apl}{W} = \frac{W_{rod}}{W_{end}}
 \end{aligned}$$



9-11

From Prob. 9-10 $\tan \frac{\omega l}{c} = \frac{(\frac{m_{rod}}{M})}{\frac{\omega l}{c}}$



Let freq. eq. be $\omega_1 = n_1 \sqrt{\frac{Eg}{\rho_w}}$
 where n_1 depends on end cond.
 ρ_w = weight density
 ρ = mass "

$$\omega_1 = n_1 \sqrt{\frac{EA}{l} \frac{gl}{\rho_w Al}} = n_1 l \sqrt{\frac{EA}{l} \frac{g}{\rho_w Al}} = n_1 l \sqrt{\frac{k}{m_{rod}}}$$

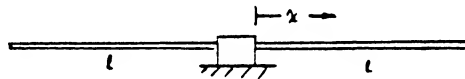
Let $n = \frac{m_{rod}}{M}$, then $\omega_1 = n_1 l \sqrt{\frac{k}{nM}}$

Approx sol.

$$\omega_{app.} \approx \sqrt{\frac{AE/l}{M + \frac{1}{3}m_{rod}}} = \sqrt{\frac{k}{M + \frac{1}{3}M}}$$

$$\frac{\omega_{app.}}{\omega_1} = \sqrt{\frac{k}{M(1 + \frac{1}{3}n)}} \sqrt{\frac{nM}{k} \left(\frac{1}{n_1 l}\right)} = \frac{1}{\beta_1} \sqrt{\frac{3n}{3+n}}$$

9-12



$$u(0) = 0 \quad \therefore u(x) = A \sin \omega \frac{x}{c}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = 0 \quad \therefore \cos \frac{\omega l}{c} = 0 \quad \frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots$$

$$\omega_1 = \frac{\pi}{l} \frac{c}{2} = \frac{\pi}{2l} \sqrt{\frac{Eg}{\rho_w}} = 2\pi (20,000)$$

$$l = \frac{1}{4 \times 20,000} \sqrt{\frac{Eg}{\rho_w}} = \frac{10^3}{80,000} \sqrt{\frac{30 \times 386}{0.31}} = 7.64'$$

Note $\rho = \frac{\rho_w}{g}$ = mass density

9-13

$$m \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial u}{\partial t} + \frac{P_0}{L} p(x) f(t)$$

$$u = \sum_i \phi_i(x) q_i(t) \quad \text{subst into eq. above}$$

$$m \sum_i \phi_i \ddot{q}_i = AE \sum_i \phi_i'' q_i - \alpha \sum_i \phi_i \dot{q}_i + \frac{P_0}{L} p(x) f(t)$$

mult. by $\phi_j dx$ and integrate over $x=0$ to l

$$m \int_0^l \phi_j \sum_i \phi_i \ddot{q}_i dx = AE \int_0^l \phi_j \sum_i \phi_i'' q_i dx - \alpha \int_0^l \phi_j \sum_i \phi_i \dot{q}_i dx + \frac{P_0}{L} \int_0^l p(x) \phi_j dx f(t)$$

Since ϕ_j and ϕ_i are orthogonal

$$\ddot{q}_j \int_0^l m \phi_j^2 dx = AE q_j \int_0^l \phi_j \phi_j'' dx - \alpha \dot{q}_j \int_0^l \phi_j^2 dx + \frac{P_0}{L} f(t) \int_0^l p(x) \phi_j dx$$

$$\ddot{q}_j + 25 \omega_j \dot{q}_j + \omega_j^2 q_j = \frac{P_0}{mL} f(t) \int_0^l p(x) \phi_j dx$$

$$\text{with } b_j = \frac{1}{L} \int_0^l p(x) \phi_j dx$$

$$\ddot{q}_j + 25 \omega_j \dot{q}_j + \omega_j^2 q_j = \frac{P_0}{m} b_j f(t)$$

From Eq. 9.2-1

$$q_j = \frac{P_0}{m} b_j \int_0^t f(t-\tau) e^{-5\omega_j \tau} \sin \omega_j \sqrt{1-5^2} \tau d\tau$$

$$u = \sum_j \phi_j q_j$$

9-14

From Eq. 9.3-3

$$c = \sqrt{\frac{2G}{\rho_w}}$$

for steel $G = 12 \times 10^6 \text{ psi}$

$$\rho_w = 0.282 \text{ lb/in}^3$$

$$c = 10^3 \sqrt{\frac{12 \times 386}{0.282}} = 128,000 \text{ in/sec} = 10,660 \text{ ft/sec}$$

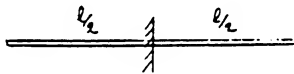
9-15

$$\theta = A \sin \frac{\omega x}{c}$$

$$\left(\frac{\partial \theta}{\partial x} \right)_{x=\frac{l}{2}} = 0 \quad \therefore \cos \frac{\omega l}{2c} = 0$$

$$\frac{\omega l}{2c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots$$

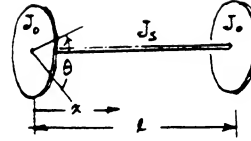
$$\omega_n = \frac{\pi c}{l}, \frac{3\pi c}{l}, \dots = (2n-1) \frac{c}{l} \quad n=1, 2, 3 \dots$$



9-16

$$J_0 \ddot{\theta}_{x=0} = G I_p \left(\frac{d\theta}{dx} \right)_{x=0}$$

$$J_0 \ddot{\theta}_{x=l} = -G I_p \left(\frac{d\theta}{dx} \right)_{x=l}$$



$$\theta = (A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}) \sin \omega t$$

$$-\omega^2 J_0 B = G I_p \frac{\omega}{c} A$$

$$-\omega^2 J_0 [A \sin \frac{\omega l}{c} + B \cos \frac{\omega l}{c}] = -G I_p \frac{\omega}{c} [A \cos \frac{\omega l}{c} - B \sin \frac{\omega l}{c}]$$

$$\therefore B = -\frac{G I_p}{\omega c J_0} A$$

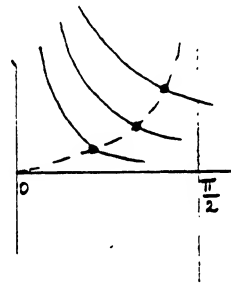
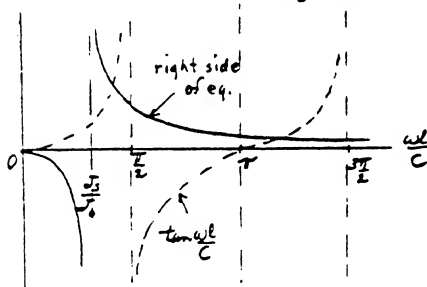
$$\sin \frac{\omega l}{c} - \frac{G I_p}{\omega c J_0} \cos \frac{\omega l}{c} = \frac{G I_p}{\omega c J_0} (\cos \frac{\omega l}{c} + \frac{G I_p}{\omega c J_0} \sin \frac{\omega l}{c})$$

$$\frac{G I_p}{\omega c J_0} = \frac{G I_p}{\omega c J_0} \frac{\rho l}{\rho l} \frac{g}{\rho l} = \frac{G g}{\rho} \frac{J_s}{J_0} \frac{1}{\omega c} = \frac{J_s}{J_0} \frac{c}{\omega l}$$

$$\therefore \left\{ \tan \frac{\omega l}{c} \right\} \left\{ 1 + \left(\frac{J_s}{J_0} \frac{c}{\omega l} \right)^2 \right\} = 2 \frac{J_s}{J_0} \frac{c}{\omega l}$$

$$\tan \frac{\omega l}{c} = \frac{2 \left(\frac{J_s}{J_0} \frac{\omega l}{c} \right)}{\left(\frac{J_s}{J_0} \frac{\omega l}{c} \right)^2 - 1}$$

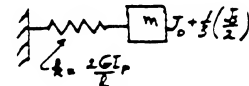
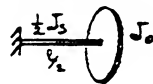
$$\frac{J_s}{J_0} < 1$$



When ends are free ($J_0 = 0$), $\frac{\omega l}{c} = \frac{\pi}{2}$. The effect of J_0 is to lower nat. freq. If J_0/J_s is very large the right side of eq

$$\approx \frac{2}{\left(\frac{J_s}{J_0} \frac{\omega l}{c} \right)}$$

Fundamental freq. has node at center



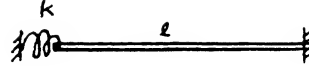
$$\omega_1 \approx \sqrt{\frac{2G I_p / l}{J_0 + \frac{1}{2} J_s}} = \sqrt{\frac{2G g}{\rho} \frac{J_s / l^2}{J_0 + \frac{1}{2} J_s}} = \frac{c}{l} \sqrt{\frac{2 J_s}{J_0 + \frac{1}{2} J_s}} = \frac{c}{l} \sqrt{\frac{2}{\frac{J_0}{J_s} + \frac{1}{2}}}$$

$$\text{When } \frac{J_0}{J_s} = 0 \quad \frac{\omega_1 l}{c} = \sqrt{3}$$

$$\text{When } \frac{J_0}{J_s} = 5, \text{ exact eq } \tan \frac{\omega l}{c} = \frac{10 \frac{\omega l}{c}}{25 \left(\frac{\omega l}{c} \right)^2 - 1} \text{ has root } \frac{\omega_1 l}{c} = 0.622$$

$$\text{Approx eq. } \frac{\omega_1 l}{c} = \sqrt{\frac{2}{5 + \frac{1}{2}}} \text{ gives } 0.62$$

9-17



$$\theta = (A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}) \sin \omega t$$

$$\theta(l) = 0 \quad \therefore A \sin \frac{\omega l}{c} + B \cos \frac{\omega l}{c} = 0, \quad B = -A \tan \frac{\omega l}{c}$$

$$\theta = A (\sin \frac{\omega x}{c} - \tan \frac{\omega l}{c} \cos \frac{\omega x}{c}) \sin \omega t$$

$$\text{Torque at } x=0 = K \theta(0) = K A (-\tan \frac{\omega l}{c})$$

$$\left(\frac{d\theta}{dx} \right)_{x=0} = A \frac{\omega}{c} (1)$$

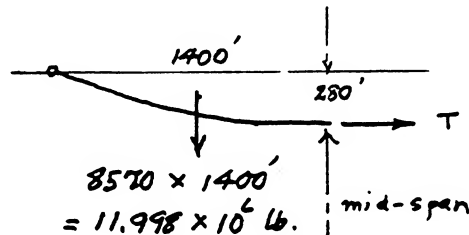
$$G I_p \left(\frac{d\theta}{dx} \right)_{x=0} = A \frac{\omega}{c} G I_p = -K A \tan \frac{\omega l}{c}$$

$$\therefore \tan \frac{\omega l}{c} = - \frac{\omega I_p G}{K c} = - \frac{I_p G}{K l} \left(\frac{\omega l}{c} \right)$$

9-18

- (a) Cables are non-horizontal
- (b) Cable tension varies along length
- (c) Stiffness of floor neglected
- (d) Torsional mass moment of inertia greatly simplified

9-19



$$11.998 \times 10^6 \times 700' = 280T, \quad T = 29.995 \times 10^6 \text{ lb.}$$

$$K_T = T b^2 = 29.995 \times 10^6 \times 60^2 = 107,980 \times 10^6 \text{ lb ft}^2/\text{rad.}$$

$$\rho = \text{density} = \frac{8570}{32.2} = 266.1$$

Transverse vibration (see Ex. 9.4-1)

$$f_n = \frac{n}{2l} \sqrt{\frac{T}{\rho}} = \frac{n}{2 \times 2800} \sqrt{\frac{29.99 \times 10^6}{266.1}} = \frac{335.7}{5600} \times n$$

$$= 0.0600 n = 3.59 \text{ c.p.m.} = f_1$$

$$f_2 = 7.19 \text{ c.p.m.}$$

9-19 Cont. Torsional vibration with node at mid-span ($n=2$)

$$\text{Estimate } J = \frac{8570}{32.2} \times K^2$$

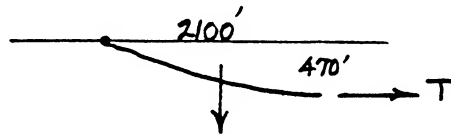
$$K = \text{rad. of gyr. (assumed)} \cdot 73 \times 30' = 22'$$

$$\therefore J = 128,816$$

$$f_2 = \frac{1}{l} \sqrt{\frac{Tb^2}{J}} \quad \gamma_2 = l \sqrt{\frac{J}{Tb^2}} = 2800 \sqrt{\frac{128,816}{107,982}} \times 10^{-3}$$

$$= 3.06 \text{ sec.} \quad f_2 = 0.327$$

9-20



$$T = \frac{28,720 \times 2100}{470'} = \frac{60.312 \times 10^6 \times \frac{1}{2}(2100)}{470'} = 134.7 \times 10^6$$

$$Tb^2 = 134.7 \times 90^2 \times 10^6 = 1091.4 \times 10^6$$

= about 10 times torsional stiffness of New Tacoma bridge.

9-21

Rewrite equation as

$$J_0 \ddot{\theta} + [c\dot{\theta} - f_2(v, \dot{\theta})] + [k\theta - f_1(v, \theta)] = 0$$

Depending on f_1 and f_2 , the middle term may become negative $-c_{eq}\dot{\theta}$ creating a possibility of negative damping. The problem requires experimental research to determine f_1 and f_2 , which can be very difficult.

9-22

$y = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x$
 At $x=0$ and $x=l$, $\frac{d^2 y}{dx^2} = \frac{d^3 y}{dx^3} = 0$ resulting in four boundary equations

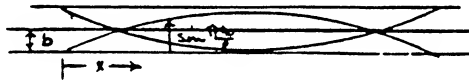
$$\text{at } x=0 \begin{cases} A + 0 - C + 0 = 0 \\ 0 + B + 0 - D = 0 \end{cases} \therefore C = A \quad D = B$$

$$\text{at } x=l \begin{cases} A \cosh \beta l + B \sinh \beta l - C \cos \beta l - D \sin \beta l = 0 \\ A \sinh \beta l + B \cosh \beta l + C \sin \beta l - D \cos \beta l = 0 \end{cases}$$

$$-\frac{A}{B} = \frac{\sinh \beta l - \sin \beta l}{\cosh \beta l - \cos \beta l} = \frac{\cosh \beta l - \cos \beta l}{\sinh \beta l + \sin \beta l}$$

$$\therefore \text{freq. eq. becomes } \underline{\underline{\cosh \beta l \cos \beta l = 1.0}}$$

9-23



Momentum =

$$m \int_0^l (\sin \frac{\pi x}{l} - b) dx = 0 \quad \text{gives } b = \frac{2}{\pi}$$

$$T = \frac{1}{2} m \omega^2 \int_0^l (\sin \frac{\pi x}{l} - b)^2 dx = \frac{1}{2} m \omega^2 \left[\frac{l}{2} - \frac{8}{\pi^2} l + \frac{4}{\pi^2} l \right] = \frac{1}{2} m \omega^2 l \left(\frac{1}{2} - \frac{4}{\pi^2} \right)$$

$$U = \frac{1}{2} EI \int_0^l \left(\frac{dy}{dx} \right)^2 dx = \frac{1}{2} EI \left(\frac{\pi}{l} \right)^2 \int_0^l \left(1 - \cos 2 \frac{\pi x}{l} \right) dx = \frac{1}{2} EI \left(\frac{\pi}{l} \right)^2 \frac{l}{2}$$

$$\text{Equate } T \text{ \& } U \quad \omega_1^2 = \frac{\pi^4}{\pi^2 - 8} \left(\frac{EI}{m l^3} \right) = 5.12 \left(\frac{EI}{m l^3} \right)$$

$$\omega_1 = 24.6 \sqrt{\frac{EI}{m l^3}} \quad \text{node at } \left(\sin \frac{\pi x}{l} - \frac{2}{\pi} \right) = 0 \quad \text{or } \frac{x}{l} = 0.22$$

9-24

$$2\pi f_1 = 22.4 \sqrt{\frac{EI}{m l^3}} = 2\pi 1690$$

$$m = 2 \times 2 \times 1 \times \frac{153}{1732} \times \frac{1}{386} = 916 \times 10^{-6}$$

$$I = \frac{2 \times 2^3}{12} = \frac{8}{3} \quad \frac{EI}{m l^3} = \left(\frac{2\pi \times 1690}{22.4} \right)^2 = 224,000$$

$$E = \frac{224,000 \times 916 \times 10^{-6} \times 12^3}{4/3} = 3,480,000 \text{ lb/in}^2$$

9-25

Start with Eq. 9.5-12, At $x=0$ and $x=l$, $y = \frac{dy}{dx} = 0$

$$\left. \begin{aligned} A + 0 + C + 0 &= 0 \\ 0 + B + 0 + D &= 0 \end{aligned} \right\} \begin{aligned} C &= -A \\ D &= -B \end{aligned}$$

$$A(\cosh \beta l - \cosh \beta l) + B(\sinh \beta l - \sinh \beta l) = 0$$

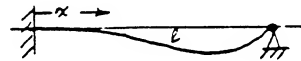
$$A(\sinh \beta l + \sinh \beta l) + B(\cosh \beta l - \cosh \beta l) = 0$$

$$-\frac{A}{B} = \frac{\sinh \beta l - \sinh \beta l}{\cosh \beta l - \cosh \beta l} = \frac{\cosh \beta l - \cosh \beta l}{\sinh \beta l + \sinh \beta l}$$

$$\text{or } \cosh \beta l \cdot \cosh \beta l = 1$$

9-26

Start with Eq. 9.5-12



$$\text{At } x=0, \quad y = \frac{dy}{dx} = 0 \quad \text{gives} \quad \begin{aligned} C &= -A \\ D &= -B \end{aligned}$$

$$\text{At } x=l \quad y = \frac{d^2y}{dx^2} = 0$$

$$A(\cosh \beta l - \cosh \beta l) + B(\sinh \beta l - \sinh \beta l) = 0$$

$$A(\cosh \beta l + \cosh \beta l) + B(\sinh \beta l + \sinh \beta l) = 0$$

$$-\frac{A}{B} = \frac{\sinh \beta l - \sinh \beta l}{\cosh \beta l - \cosh \beta l} = \frac{\sinh \beta l + \sinh \beta l}{\cosh \beta l + \cosh \beta l}$$

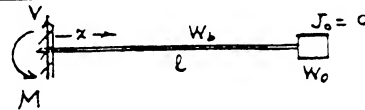
$$\therefore \cosh \beta l \sinh \beta l - \sinh \beta l \cosh \beta l = 0$$

$$\text{or } \tanh \beta l = \tanh \beta l$$

9-27

$$\text{At } x=0, \quad y = \frac{dy}{dx} = 0$$

$$\therefore C = -A, \quad D = -B$$



$$\text{At } x=l \quad -EI \frac{d^3y}{dx^3} = -V = \frac{W_0}{g} \ddot{y}(l)$$

$$\begin{aligned} -\beta^3 [A(\sinh \beta l - \sinh \beta l) + B(\cosh \beta l + \cosh \beta l)] \\ = -\frac{W_0}{g} \frac{\omega^2}{EI} [A(\cosh \beta l - \cosh \beta l) + B(\sinh \beta l - \sinh \beta l)] \end{aligned}$$

9-27 Cont.

$$\text{At } x=l \quad -M = -EI \frac{d^2 y}{dx^2} = J_0 \left(\frac{dy}{dx} \right) = 0$$

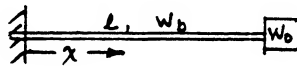
$$\beta^2 [A (\cos \beta l + \cosh \beta l) + B (\sinh \beta l + \sin \beta l)] = 0$$

$$-\frac{A}{B} = \frac{(\cos \beta l + \cosh \beta l) - \frac{W_0 \omega^2}{\beta^2 g EI} (\sinh \beta l - \sin \beta l)}{(\sinh \beta l - \sin \beta l) - \frac{W_0 \omega^2}{\beta^2 g EI} (\cos \beta l - \cosh \beta l)} = \frac{(\sinh \beta l + \sin \beta l)}{(\cosh \beta l + \cos \beta l)}$$

$$\beta^2 = \omega \sqrt{\frac{W l}{g EI l}} = \omega \sqrt{\frac{W_b}{g EI l}}$$

$$\therefore (1 - \cosh \beta l \cos \beta l) = \frac{W_0}{W_b} \beta l (-\cosh \beta l \sin \beta l + \sinh \beta l \cos \beta l)$$

9-28



Assume static curve for concentrated load at end.

$$y(x) = \frac{1}{EI} \left(\frac{l x^2}{2} - \frac{x^3}{6} \right) \quad \text{Effective mass } m_{\text{eff}} \text{ of uniform beam at end of beam is}$$

$$T = \frac{1}{2} \int_0^l y^2 dm = \frac{1}{2} m_{\text{eff}} \left(\frac{l^2}{3EI} \right)^2, \quad T = \frac{1}{2(EI)^2} \int_0^l \left(\frac{l x^2}{2} - \frac{x^3}{6} \right)^2 dx$$

$$\therefore m_{\text{eff}} = 9 \times 0.262 ml = .235 ml \quad = \frac{1}{2} \frac{ml}{(EI)^2} \cdot 0.262 l^4$$

$$\text{Total eff. mass} = \frac{1}{g} (.235 W_b + W_0), \quad \omega_1 = \sqrt{\frac{3EIg}{l^3 (.235 W_b + W_0)}}$$

9-29

Additional mass to those of Prob 9-28 is

$$W_1 \text{ at } x = x_1, \quad \text{The KE of } W_1 \text{ is } \frac{1}{2} \frac{W_1}{g} y(x_1)^2$$

$$\therefore \text{Total effective mass is } \frac{1}{g} \left[.235 W_b + W_0 + W_1 \left(\frac{y(x_1)}{y(l)} \right)^2 \right]$$

and

$$\omega_1 = \sqrt{\frac{3EIg}{[.235 W_b + \left(\frac{y(x_1)}{y(l)} \right)^2 W_1 + W_0]}}$$

9-30

$$\text{At } x=0, y=y_0 \therefore y_0 = A + C$$

$$\text{At } x=0 \quad \frac{d^2 y}{dx^2} = 0 \therefore C = A$$

$$\text{At } x=l \quad \frac{d^2 y}{dx^2} = 0$$

$$\beta^2 [A(\cosh \beta l - \cos \beta l) + B \sinh \beta l - D \sin \beta l] = 0$$

$$\text{At } x=l \quad \frac{d^3 y}{dx^3} = 0$$

$$\beta^3 [A(\sinh \beta l + \sin \beta l) + B \cosh \beta l - D \cos \beta l] = 0$$

$$\text{At } x=l, y=y_l$$

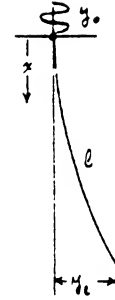
$$y_l = A(\cosh \beta l + \cos \beta l) + B \sinh \beta l + D \sin \beta l$$

$$\frac{y_0}{y_l} = \frac{2A}{A(\cosh \beta l + \cos \beta l) + B \sinh \beta l + D \sin \beta l} = \frac{2A}{A(\cosh \beta l + \cos \beta l) + B \sinh \beta l + A(\cosh \beta l - \cos \beta l) + B \sinh \beta l}$$

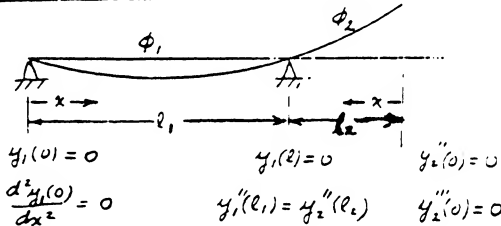
$$\text{where } Ch = \cosh \beta l, C = \cos \beta l, Sh = \sinh \beta l, S = \sin \beta l$$

$$\text{Also } y_0 = 2A$$

$$\frac{y_0}{y_l} = \frac{\sinh \beta l \cos \beta l - \cosh \beta l \sin \beta l}{\sinh \beta l - \sin \beta l}$$



9-31



$$\begin{array}{lll} \text{Boundary} & y_1(0) = 0 & y_1(l_1) = 0 \quad y_2''(0) = 0 \\ \text{intermediate cond.} & \frac{d^2 y_1(0)}{dx^2} = 0 & y_1'(l_1) = y_2''(l_1) \quad y_2'''(0) = 0 \end{array}$$

$$\left. \begin{array}{l} y_1(0) = A_1 + C_1 = 0 \quad \therefore C_1 = -A_1 \\ y_1''(0) = A_1 - C_1 = 0 \quad \therefore C_1 = +A_1 \end{array} \right\} \therefore A_1 = C_1 = 0$$

$$y_1 = B_1 \sinh \beta x + D_1 \sin \beta x$$

$$y_1(l_1) = B_1 \sinh \beta l_1 + D_1 \sin \beta l_1 = 0 \quad \therefore B_1 = -D_1 \frac{\sin \beta l_1}{\sinh \beta l_1}$$

$$y_1 = D_1 \left(\sin \beta x - \frac{\sin \beta l_1}{\sinh \beta l_1} \sinh \beta x \right) = \phi_1(x)$$

$$y_2''(0) = A_2 - C_2 = 0 \quad \therefore C_2 = A_2$$

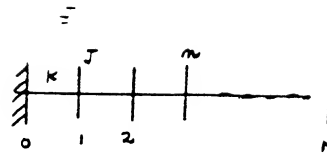
$$y_2'''(0) = B_2 - D_2 = 0 \quad \therefore D_2 = B_2$$

$$y_2(l_1) = A_2(\cosh \beta l_1 + \cos \beta l_1) + B_2(\sinh \beta l_1 + \sin \beta l_1) = 0$$

$$\therefore B_2 = -A_2 \frac{\cosh \beta l_1 + \cos \beta l_1}{\sinh \beta l_1 + \sin \beta l_1}$$

$$y_2 = A_2 \left\{ (\cosh \beta x + \cos \beta x) - \left(\frac{\cosh \beta l_1 + \cos \beta l_1}{\sinh \beta l_1 + \sin \beta l_1} \right) (\sinh \beta x + \sin \beta x) \right\} = \phi_2(x)$$

9-32



$$-J\omega^2 \theta_n = K(\theta_{n+1} - \theta_n) - K(\theta_n - \theta_{n-1})$$

$$\theta_{n-1} - 2\left(1 - \frac{\omega^2 J}{2K}\right)\theta_n + \theta_{n+1} = 0 \quad \text{let } \theta_n = e^{\lambda n}$$

$$e^{-\lambda} - 2\left(1 - \frac{\omega^2 J}{2K}\right) + e^{\lambda} = 0$$

$$\therefore \frac{\omega^2 J}{K} = 2(1 - \cosh \lambda)$$

$$\lambda = i\beta$$

$$= 2(1 - \cos \beta)$$

$$\theta_n = e^{i\beta n} = \cos \beta n + i \sin \beta n$$

Assume sol. $\theta_n = A \sin \beta n + B \cos \beta n$

Boundary $\theta_0 = 0 \therefore B = 0$ and $\theta_n = A \sin \beta n$

Boundary N $-\omega^2 J \theta_N = -K(\theta_N - \theta_{N-1})$

$$\left(1 - \frac{\omega^2 J}{K}\right)\theta_N = \theta_{N-1}$$

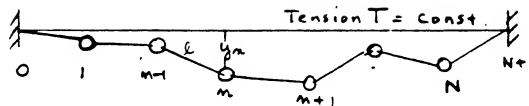
$$[1 - 2(1 - \cos \beta)] \sin \beta N = \sin \beta(N-1)$$

Reduces to $2 \sin \beta N \cos \beta = \sin \beta(N-1) + \sin \beta N$

$$2 \cos \beta \left(N + \frac{1}{2}\right) \sin \frac{\beta}{2} = 0 \therefore \beta = 0, 2\pi, \dots$$

$$\omega_k = 2 \sqrt{\frac{K}{J}} \sin \frac{(2k-1)\pi}{2(2N+1)} \quad k = 1, 2, 3, \dots, N$$

9-33



$$m \ddot{y}_m = -\frac{T}{l} (y_m - y_{m-1}) + \frac{T}{l} (y_{m+1} - y_m) \quad \text{rearrange to}$$

$$y_{m+1} - 2 \left(1 - \frac{\omega^2 m l}{2T} \right) y_m + y_{m-1} = 0$$

let $y_m = e^{i\beta m}$ + subst. into above eq.

$$\cancel{e^{i\beta m}} e^{i\beta} - 2 \left(1 - \frac{\omega^2 m l}{2T} \right) \cancel{e^{i\beta m}} + \cancel{e^{i\beta m}} e^{-i\beta} = 0$$

$$1 - \frac{\omega^2 m l}{2T} = \frac{e^{i\beta} + e^{-i\beta}}{2} = \cos \beta$$

$$\frac{\omega^2 m l}{2T} = 1 - \cos \beta = 2 \sin^2 \frac{\beta}{2}$$

freq. eq. $\omega = 2 \sqrt{\frac{T}{m l}} \sin \frac{\beta}{2}$ Evaluate β at boundaries
starting with gen. sol.

$$y_m = A \cos \beta m + B \sin \beta m$$

$$y_0 = 0 \quad \therefore A = 0, \quad y_{N+1} = 0 \quad \therefore \sin \beta(N+1) = 0$$

$$\beta(N+1) = 0, \pi, 2\pi, 3\pi, \dots = k\pi$$

$$\therefore \omega_k = 2 \sqrt{\frac{T}{m l}} \sin \frac{k\pi}{2(N+1)} \quad k = 1, 2, 3, \dots$$

$$\text{For } N=2 \quad \omega_1 = 2 \sqrt{\frac{T}{m l}} \sin \frac{\pi}{6} = \sqrt{\frac{T}{m l}}$$

$$\omega_2 = 2 \sqrt{\frac{T}{m l}} \sin \frac{2\pi}{6} = \sqrt{\frac{3T}{m l}}$$

9-34

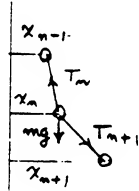
$$m \ddot{x}_m = k(x_{m+1} - x_m) - k_m(x_m - x_{m-1})$$

$$x_{m+1} - 2 \left(1 - \frac{\omega^2 m}{k} \right) x_m + x_{m-1} = 0$$

This problem is identical to Prob 9-3. with **changes** in
boundary conditions $\therefore \omega_k = 2 \sqrt{\frac{k}{m}} \sin \frac{k\pi}{2(N+1)}$ $k=1, 2, \dots$

9-35

$$m\ddot{x}_m = \frac{T_{m+1}}{l}(x_{m+1} - x_m) - \frac{T_m}{l}(x_m - x_{m-1})$$



$$T_m \approx T'_{m+1} + mlg$$

$$x_{m+1} - 2(1 - \frac{\omega^2 m l}{T_m})x_m + x_{m-1} = \frac{mgl}{T_m}(x_{m+1} - x_m)$$

T_m changes i.e. $T_N = mg$, $T_{N-1} = 2mg$ etc.

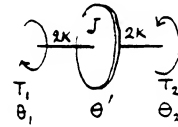
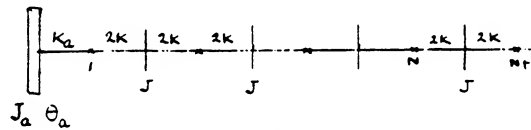
∴ Usual diff eq with const. coef. cannot be used.

Must solve as a system of N eqs.

$$-\omega^2 m x_1 = -\frac{Nmg}{l}x_1 + (N-1)\frac{mg}{l}(x_2 - x_1)$$

$$-\omega^2 m x_N = -\frac{mg}{l}(x_N - x_{N-1})$$

9-36



$$T_1 = 2k(\theta' - \theta_1) \quad \rightarrow \quad \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2k} & 0 \\ -\frac{1}{2k} & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ \theta' \end{Bmatrix}$$

$$T_2 - T_1 = -\omega^2 J \theta' \quad \rightarrow \quad \begin{Bmatrix} T_1 \\ \theta' \end{Bmatrix} = \begin{bmatrix} 1 & \omega^2 J \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta' \end{Bmatrix}$$

$$T_2 = 2k(\theta_2 - \theta') \quad \rightarrow \quad \begin{Bmatrix} T_2 \\ \theta' \end{Bmatrix} = \begin{bmatrix} \frac{1}{2k} & 0 \\ -\frac{1}{2k} & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2k} & 0 \\ -\frac{1}{2k} & 1 \end{bmatrix} \begin{bmatrix} 1 & \omega^2 J \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2k} & 0 \\ -\frac{1}{2k} & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} (1 - \frac{\omega^2 J}{2k}) & \omega^2 J & \frac{1}{2k} \\ -(\frac{1}{k} - \frac{\omega^2 J}{2k}) & (-\frac{\omega^2 J}{2k}) & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & A \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix}$$

9-36 Cont.

Note that $A^2 - BC = 1$

$$B = \frac{A^2 - 1}{C}$$

$$\text{Let } A = \left(1 - \frac{\omega^2 J}{2K}\right) = \cosh \lambda$$

$$C = -\left(\frac{1}{K} - \frac{\omega^2 J}{4K}\right) = \frac{1}{2} \sinh \lambda$$

$$B = \omega^2 J = \frac{\cosh^2 \lambda - 1}{\frac{1}{2} \sinh \lambda} = 2 \sinh \lambda$$

$$\begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \cosh \lambda & 2 \sinh \lambda \\ \frac{1}{2} \sinh \lambda & \cosh \lambda \end{bmatrix} \begin{Bmatrix} T_2 \\ \theta_2 \end{Bmatrix} \quad \begin{array}{l} \text{If replaced by} \\ \text{exponentials it can be} \\ \text{proved that} \end{array}$$

$$\begin{bmatrix} \cosh \lambda & 2 \sinh \lambda \\ \frac{1}{2} \sinh \lambda & \cosh \lambda \end{bmatrix}^n = \begin{bmatrix} \cosh n\lambda & 2 \sinh n\lambda \\ \frac{1}{2} \sinh n\lambda & \cosh n\lambda \end{bmatrix}$$

$$\therefore \begin{Bmatrix} T_1 \\ \theta_1 \end{Bmatrix} = \begin{bmatrix} \cosh n\lambda & 2 \sinh n\lambda \\ \frac{1}{2} \sinh n\lambda & \cosh n\lambda \end{bmatrix} \begin{Bmatrix} T_{n+1} \\ \theta_{n+1} \end{Bmatrix}$$

$$\begin{array}{l} \text{From } A = \left(1 - \frac{\omega^2 J}{2K}\right) = \cosh \lambda \\ \omega^2 J = B = 2K(1 - \cosh \lambda) = -4K \sinh^2 \frac{\lambda}{2} \end{array} \quad \left\{ \begin{array}{l} \therefore \text{freq. eq. } \omega = 2 \sqrt{\frac{K}{J}} \sinh \frac{\lambda}{2} \\ \text{where } \lambda \text{ must be determined from boundary} \end{array} \right.$$

Boundary conditions

$$\theta_1 - \theta_a = \frac{T_1}{K_a} = -\frac{\omega^2 J_a \theta_a}{K_a}, \quad T_{N+1} = 0$$

$$\therefore T_1 = 2 \sinh N\lambda \cdot \theta_{N+1} = -\omega^2 J_a \theta_a$$

$$\theta_1 = \cosh N\lambda \cdot \theta_{N+1} = \left(1 - \frac{\omega^2 J_a}{K_a}\right) \theta_a$$

$$\text{Divide } \frac{2 \sinh N\lambda}{\cosh N\lambda} = \frac{-\omega^2 J_a}{\left(1 - \frac{\omega^2 J_a}{K_a}\right)}$$

$$\frac{2K(1 - \cosh \lambda)}{\sinh \lambda} \cdot \frac{\sinh N\lambda}{\cosh N\lambda} = \frac{-\omega^2 J_a}{\left(1 - \frac{\omega^2 J_a}{K_a}\right)}$$

$$-2K \left[2 \sinh N\lambda \cosh \lambda - \sinh N\lambda \right] = \frac{4K \frac{J_a}{J} \sinh^2 \frac{\lambda}{2}}{1 + \frac{4K J_a}{K_a J} \sinh^2 \frac{\lambda}{2}} \cdot \sinh \lambda \cosh N\lambda$$

$$(-2 \sinh N\lambda \cdot \cosh \lambda + 2 \sinh N\lambda) \left(1 - \frac{4K}{K_a} \frac{J_a}{J} \sinh^2 \frac{\lambda}{2}\right) = 2 \frac{J_a}{J} \sinh^2 \frac{\lambda}{2} \sinh \lambda \cosh N\lambda$$

$$(-\sin N\beta \cos \beta + \sin N\beta) \left(1 + 4 \frac{K}{K_a} \frac{J_a}{J} \sin^2 \frac{\beta}{2}\right) = -2 \frac{J_a}{J} \sin^2 \frac{\beta}{2} \sin \beta \cos N\beta$$

Solve for β & subst. into freq. eq

$$\omega = 2 \sqrt{\frac{K}{J}} \sin \frac{\beta}{2}$$

This problem can also be solved by using stations at the disks instead of at points midway between disks. The equations

$$\theta_n = A \sin \rho n + B \cos \rho n, \quad n = 0, 1, 2, \dots, N$$

can be used provided $K_0 = \frac{1}{\frac{1}{K_a} + \frac{1}{2K}}$ is used.

9-37

With $m = 0$ fixed, solution is

$$X_m = B \sin \beta m$$

At top of the bldg, $m \ddot{x}_N = -k(x_N - x_{N-1}) - K_N x_N$

$$\text{or } X_{N-1} = \left(1 + \frac{K_N}{k} - \frac{m\omega^2}{k}\right) X_N$$

$$\therefore \sin \beta(N-1) = \left[1 + \frac{K_N}{k} - 2(1 - \cos \beta)\right] \sin \beta N$$

$$\sin \beta(N-1) - \sin \beta N + 2(1 - \cos \beta) \sin \beta N = \frac{K_N}{k} \sin \beta N$$

$$\sin \beta N \cos \beta - \cos \beta N \sin \beta - \sin \beta N + 2 \sin \beta N - 2 \sin \beta N \cos \beta = \frac{K_N}{k} \sin \beta N$$

$$-\sin \beta N \cos \beta - \cos \beta N \sin \beta + \sin \beta N = \frac{K_N}{k} \sin \beta N$$

$$\therefore -\sin \beta(N+1) + \sin \beta N = \frac{K_N}{k} \sin \beta N$$

$$2 \cos \beta(N + \frac{1}{2}) \cdot \sin \frac{\beta}{2} = 2 [\cos \beta N \cos \frac{\beta}{2} \sin \frac{\beta}{2} - \sin \beta N \sin \frac{\beta}{2}] = [\cos \beta N \sin \beta - \sin \beta N (1 - \cos \beta)]$$

$$= \cos \beta N \sin \beta - \sin \beta N + \sin \beta N \cos \beta$$

$$\therefore -2 \cos \beta(N + \frac{1}{2}) \cdot \sin \frac{\beta}{2} = \frac{K_N}{k} \sin \beta N$$

9-38

System is same as those of FIG. P9-33 & P9-34

$$\therefore \omega_m = 2 \sqrt{\frac{k}{m}} \sin \frac{m\pi}{2(N+1)}$$

9-39

Let h = height between stories

$$m \ddot{y}_n = -k[(y_n - h\theta) - (y_{n-1} - (n-1)h\theta)] + k[(y_{n+1} - (n+1)h\theta) - (y_n - nh\theta)]$$

$$= -k[y_n - y_{n-1} - h\theta] + k[y_{n+1} - y_n - h\theta]$$

$$= k[y_{n+1} - 2y_n + y_{n-1}]$$

\therefore for harmonic motion

$$Y_{n+1} - 2\left(1 - \frac{m\omega^2}{2k}\right)Y_n + Y_{n-1} = 0$$

gen. sol. $Y_n = Y_0 \cos \beta n + B \sin \beta n$

boundary $Y_N = Y_0 \cos \beta N + B \sin \beta N$

$$B = \frac{Y_N - Y_0 \cos \beta N}{\sin \beta N}$$

Gen sol. becomes

$$Y_n = Y_0 \frac{\sin \beta(N-n)}{\sin \beta N} + Y_N \frac{\sin \beta n}{\sin \beta N}$$

boundary eq. for N^{th} mass

$$m \ddot{y}_N = -k[y_N - y_{N-1} + h\theta]$$

$$-\omega^2 m Y_N = -k[Y_N - Y_{N-1} + h\theta]$$

9-9 Cont.

$$\underline{\underline{\left(1 - \frac{\omega^2 m}{k}\right) Y_N = Y_{N-1} - l \theta}}$$

Torque Eq

$$\sum_{n=1}^N m l (m \ddot{y}_N) - k_\theta \theta = (N+1) m \rho^2 \ddot{\theta}$$

$$-\omega^2 m l \sum_{n=1}^N m Y_n - k_\theta \theta = -(N+1) m \rho^2 \omega^2 \theta$$

or

$$\underline{\underline{\omega^2 m l \sum_{n=1}^N m Y_n - (k_\theta - (N+1) m \rho^2 \omega^2) \theta = 0}}$$

9-40

$$\beta_k = \frac{(2k-1)\pi}{(2N+1)}$$

$$\omega_k = 2\sqrt{\frac{k}{m}} \sin \frac{(2k-1)\pi}{2(2N+1)}$$

$$\left. \begin{array}{l} N = \text{DOF} \\ k = \text{mode number} \end{array} \right\}$$

Then $X_n = B \sin \beta_k n$. where $n = \text{floor number}$

9-41

The stiffness of the cantilever beam is $\frac{3EI}{l^3}$, i.e.,

in the absence of the spring K , $F = \frac{3EI}{l^3} y$.

In the antisymmetric mode, where $y_1 = -y_2 = y$, the spring force is $K(y_1 - y_2) = 2Ky$

$$\begin{aligned}\therefore F &= \frac{3EI}{l^3} y + 2Ky \quad (\text{For each beam}) \\ &= \left(\frac{3EI}{l^3} + 2K \right) y = K_{eq} y\end{aligned}$$

In the absence of the spring force $\omega_1 = 3.52 \sqrt{\frac{EI}{\rho l^4}}$

With the spring in the antisymmetric mode

$$\begin{aligned}\omega_1 &= \sqrt{\frac{\frac{3EI}{l^3} + 2K}{\frac{3EI}{l^3}}} \times 3.52 \sqrt{\frac{EI}{\rho l^4}} \\ &= 3.52 \sqrt{\frac{EI}{\rho l^4} \left(1 + \frac{2}{3} \frac{K l^3}{EI} \right)}\end{aligned}$$

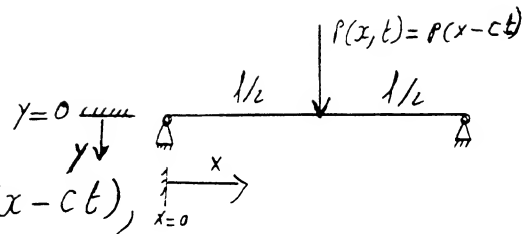
9-42

Euler equation

($y = y(x, t)$)

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = p(x, t) = p(x - ct), \quad x=0$$

$$y(0, t) = y(1, t) = 0, \quad \frac{\partial^2 y}{\partial x^2} \Big|_{x=0} = \frac{\partial^2 y}{\partial x^2} \Big|_{x=1} = 0 \quad \text{for all } t$$



$$\text{Let } y(x, t) = y(x - ct)$$

$$\therefore EI \frac{\partial^4 y(x - ct)}{\partial x^4} + m \frac{\partial^2 y(x - ct)}{\partial t^2} = p(x - ct)$$

$$\text{Let } z = x - ct$$

$$\therefore EI \frac{\partial^4 y}{\partial z^4} + mc^2 \frac{\partial^2 y}{\partial z^2} = p(z)$$

$$y(0) = y(1) = 0, \quad \frac{\partial^2 y}{\partial z^2} \Big|_{z=0} = \frac{\partial^2 y}{\partial z^2} \Big|_{z=1} = 0$$

$$\text{Let } F = \frac{\partial^2 y}{\partial z^2}$$

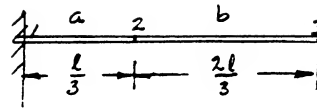
$$\therefore EIF'' + mc^2 F = p(z)$$

Solve for $F(z)$ then integrate twice to get $y(z) = y(x - ct)$.

9-43

This problem is similar to example 9.5-1. Here, the fact that the ball hits the bat, which acts as a cantilever beam, changes the boundary conditions at l . The boundary conditions do not change the natural frequencies, which are given by equation (9.5-13).

10-1 (see Ex. 10.1-1)



$$M_a = \frac{1}{3} M$$

$$k_a = \frac{EA}{l/3} = \frac{6}{2} \frac{EA}{l}$$

$$M_b = \frac{2}{3} M$$

$$k_b = \frac{EA}{2l/3} = \frac{3}{2} \frac{EA}{l}$$

with $u_1 = 0$

$$\text{Mass Matrix} = \frac{1}{6} \begin{bmatrix} 2(M_a + M_b) & M_b \\ M_b & 2M_b \end{bmatrix} \quad \text{Ex. 10.1-1}$$

$$\text{Stiffness Matrix} = \begin{bmatrix} (k_a + k_b) & -k_b \\ -k_b & k_b \end{bmatrix}$$

Diff. Eq. of Motion with $\ddot{u} = -\omega^2 u$

$$\left\{ \begin{matrix} -\omega^2 M \\ 3 \times 6 \end{matrix} \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} + \frac{EA}{2l} \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix} \right\} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 M}{18} \cdot \frac{2l}{EA} = \frac{\omega^2 M l}{9 EA}$$

Characteristic Eq.

$$\begin{vmatrix} (9-6\lambda) & -(3+2\lambda) \\ -(3+2\lambda) & (3-4\lambda) \end{vmatrix} = 0 \rightarrow \lambda^2 - 3.30\lambda + .90 = 0$$

$$\lambda = \begin{cases} .30 \\ 3.00 \end{cases}$$

$$\omega^2 = 9 \times \begin{cases} .3 \\ 3.00 \end{cases}$$

$$\omega_1 = 1.643 \sqrt{EA/Ml}$$

$$\omega_2 = 5.196 \sqrt{EA/Ml}$$

$$\text{Exact} \rightarrow \begin{cases} 1.5708 \\ 4.7124 \end{cases}$$

$\therefore 1^{st}$ mode is 4.6% high

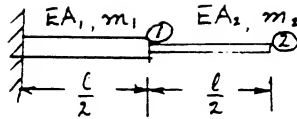
2^{nd} mode is 10.3% high

With sta. at $\frac{l}{2}$, $\lambda^2 - \frac{10}{7}\lambda + \frac{1}{7} = 0$

$$\omega_1 = 1.6115 \sqrt{\frac{EA}{Ml}} = 1.025 \times \text{Exact} = 2.6\% \text{ high}$$

$$\omega_2 = 5.6293 \sqrt{\frac{EA}{Ml}} = 1.1946 \times " = 19.5\% \text{ high}$$

10-2



$$\begin{aligned} EA_1 &= 2 EA_2 \\ m_1 &= 2 m_2 \\ k_2 &= \frac{2 EA_2}{l} \end{aligned}$$

$$\left[-\frac{\omega^2}{6} \begin{bmatrix} 2(M_a + M_b) & M_b \\ M_b & 2 M_b \end{bmatrix} + \begin{bmatrix} (k_a + k_b) & -k_b \\ -k_b & k_b \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[-\frac{\omega^2}{6} M_2 \begin{bmatrix} 2(2+1) & 1 \\ 1 & 2 \end{bmatrix} + \frac{2 EA_2}{l} \begin{bmatrix} (2+1) & -1 \\ -1 & 1 \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 M_2}{6} \left(\frac{l}{2 EA_2} \right) = \left(\frac{\omega^2 M_2 l}{12 EA_2} \right) \therefore \left| -\lambda \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (3-6\lambda) & -(1+\lambda) \\ -(1+\lambda) & (1-2\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 1.2727\lambda + 0.1818 = 0$$

$$\lambda_{1,2} = \begin{cases} 0.1640 \\ 1.1088 \end{cases}$$

$$\omega_1 = 1.4029 \sqrt{\frac{EA_2}{M_2 l}}$$

$$\omega_2 = 3.6477 \sqrt{\frac{EA_2}{M_2 l}}$$

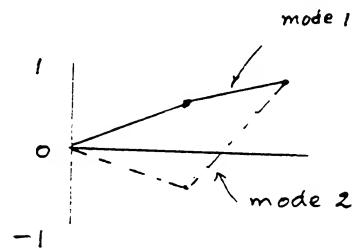
Mode shapes

$$(3-6\lambda)u_1 = (1+\lambda)u_2$$

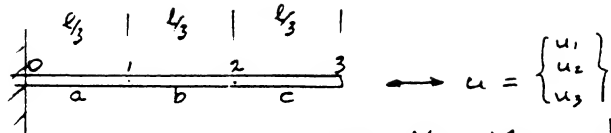
$$\text{or } (1+\lambda)u_1 = (1-2\lambda)u_2$$

$$\left(\frac{u_1}{u_2} \right)_1 = \frac{(1-2\lambda)}{(1+\lambda)} = 0.5773$$

$$\left(\frac{u_1}{u_2} \right)_2 = -0.5258$$



10-3



$$M_a = M_b = M_c = \frac{M}{3}$$

$$k_a = k_b = k_c = \frac{EA_a}{l/3} = \frac{3EA}{l}$$

Mass matrix

$$\left[\begin{array}{c} \frac{M_a}{6} \left[\begin{array}{cc} 2 & 1 \\ 1 & 2+2 \end{array} \right] \\ \frac{M_b}{6} \left[\begin{array}{cc} 2+2 & 1 \\ 1 & 2 \end{array} \right] \end{array} \right]$$

Stiffness matrix

$$k_a \left[\begin{array}{cc} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1+1 \end{array} \right] & \left[\begin{array}{c} -1 \\ 1+1 \end{array} \right] \\ \left[\begin{array}{c} -1 \\ 1+1 \end{array} \right] & \left[\begin{array}{cc} -1 & 1 \end{array} \right] \end{array} \right]$$

$$\frac{M_a}{6} \left\{ \begin{array}{c} 2 \\ 1 \\ 0 \\ 0 \end{array} \right\} \left\{ \begin{array}{c} \ddot{u}_0 \\ \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{array} \right\} + k_a \left\{ \begin{array}{c} 1 \\ -1 \\ 0 \\ 0 \end{array} \right\} \left\{ \begin{array}{c} u_0 \\ u_1 \\ u_2 \\ u_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}$$

$u_0 = 0 \therefore$ cross out as above

$$\frac{M}{18} \left\{ \begin{array}{c} 4 \\ 1 \\ 0 \end{array} \right\} \left\{ \begin{array}{c} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{array} \right\} + \frac{3EA}{l} \left\{ \begin{array}{c} 2 \\ -1 \\ 0 \end{array} \right\} \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\}$$

Computer solution: $\lambda = \frac{M\ell\omega^2}{54EA}$

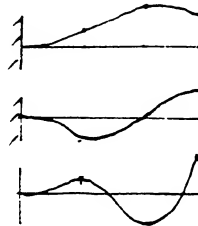
$$\omega_1 = 1.522 \sqrt{\frac{EA}{M\ell}} \quad \left(\frac{EA}{M\ell} = \frac{1}{\ell} \sqrt{\frac{E}{\rho}} \right) \quad \text{Exact } 1.571 \sqrt{\frac{E}{\rho}} \quad \therefore 3\% \text{ low}$$

$$\omega_2 = 5.1545 \text{ " } \quad 4.712 \text{ " } \quad 9\% \text{ high}$$

$$\omega_3 = 9.4329 \text{ " } \quad 7.85 \text{ " } \quad 20\% \text{ high}$$

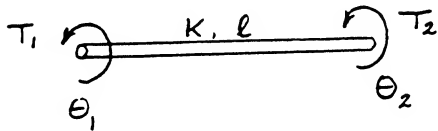
Mode Shapes

$$\left\{ \begin{array}{c} 0 \\ .415 \\ .728 \\ .544 \end{array} \right\}_1 \quad \left\{ \begin{array}{c} 0 \\ -.811 \\ .0268 \\ .583 \end{array} \right\}_2 \quad \left\{ \begin{array}{c} 0 \\ .410 \\ -.684 \\ .602 \end{array} \right\}_3$$



10-4 With linear twist, the problem is identical to that of the longitudinal vibration.

Torsional element of length l

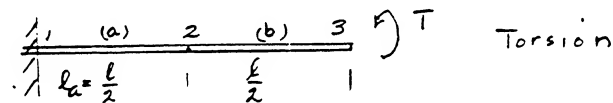


Mass = $\rho A l$ Stiffness = $\frac{I_p G}{l}$ of length l

$J_p = \frac{M R^2}{2}$ = mass polar moment of inertia

Mass matrix = $\frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Stiffness = $\frac{I_p G}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

10-5



$M_a \frac{R^2}{2} = J_a$

Let J = mass polar moment of inertia for entire rod

Then $J_a = \frac{1}{2} J$, $K_a = \frac{I_p G}{l_a} = \frac{2 I_p G}{l}$

$$\frac{J_a}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + K_a \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Units: J = mass in^2 , I_p = in^4 , G = lb/in^2 , l = in .

Let $\lambda = \frac{\omega^2 J_a}{6 K_a}$

$J_a = \frac{J}{2}$, $K_a = 2 K$

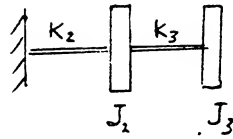
Same prob. as longitudinal vibr.

$\omega = \begin{cases} 1.6114 \sqrt{\frac{K}{J}} \\ 5.629 \text{ "} \end{cases}$ where $K = \frac{I_p G}{l}$, $J = M \frac{R^2}{2}$

$\frac{K}{J} = \frac{2 I_p G}{M l R^2} = \frac{G A}{M l}$

10-6

Lumped mass torsional system

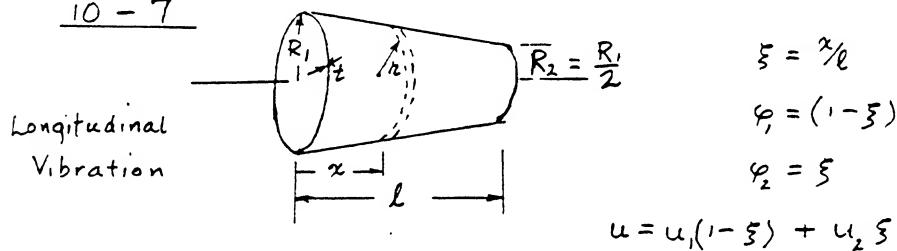


$$\begin{bmatrix} J_2 & 0 \\ 0 & J_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} (K_2 + K_3) & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} [(K_2 + K_3) - \omega^2 J_2] & -K_3 \\ -K_3 & [K_3 - \omega^2 J_3] \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \therefore \text{same form}$$

The problem here is assigning proper values of J_2 and J_3 , to be equivalent to distributed system.

10-7



$$m_{ij} = \int \varphi_i \varphi_j dm$$

$$r = R_1 (1 - \frac{1}{2} \xi)$$

$$dm = \rho 2\pi r t \cdot dx, \quad \rho = \text{mass density}$$

$$A = 2\pi r t$$

$$\therefore m_{11} = \rho 2\pi t R_1 \int_0^1 (1 - \frac{1}{2} \xi) (1 - \xi)^2 l d\xi = m_1 \int_0^1 (1 - \frac{5}{2} \xi + 2\xi^2 - \frac{1}{2} \xi^3) d\xi$$

$$\text{where } m_1 = \rho 2\pi t R_1 l = \rho A_1 l$$

$$m_{11} = m_1 (1 - \frac{5}{4} + \frac{2}{3} - \frac{1}{8}) = \frac{m_1}{24} (24 - 30 + 16 - 3) = \frac{7}{24} m_1$$

$$A_1 = 2\pi t R_1$$

10-7 Cont.

$$m_{12} = m_1 \int_0^1 (1 - \frac{1}{2}\xi)(1 - \xi)\xi d\xi = \frac{3}{24} m_1$$

$$m_{22} = m_1 \int_0^1 (1 - \frac{1}{2}\xi)\xi^2 d\xi = \frac{5}{24} m_1$$

$$\therefore m = \frac{m_1}{24} \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$$

$$k_{ij} = \int EA \varphi_i' \varphi_j' l d\xi \quad \varphi_1' = \frac{d\varphi_1}{dx} = -\frac{1}{l}, \quad \varphi_2' = \frac{1}{l}$$

$$k_{11} = E \frac{2\pi t R_1}{l^2} \int_0^1 (1 - \frac{1}{2}\xi) l d\xi = \frac{EA_1}{l} (1 - \frac{1}{4}) = \frac{3}{4} \frac{EA_1}{l}$$

$$\text{where } A_1 = 2\pi t R_1$$

$$k_{12} = -\frac{3}{4} \frac{EA_1}{l} = k_{21}, \quad k_{22} = \frac{3}{4} \frac{EA_1}{l}$$

$$\therefore k = \frac{3}{4} \frac{EA_1}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Eq. of motion

$$\frac{\rho A_1 l}{24} \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{3}{4} \frac{EA_1}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 \rho l^2}{18E}$$

$$\begin{vmatrix} (1-7\lambda) & -(1+3\lambda) \\ -(1+3\lambda) & (1-5\lambda) \end{vmatrix} = 0 \quad \therefore \lambda (26\lambda - 6) = 0$$

$$\lambda_0 = 0$$

$$\lambda_1 = \frac{6}{26} = \frac{\omega^2 \rho l^2}{18E} = \frac{\omega^2 (\rho A_1 l) l}{18EA_1} = \frac{\omega^2 m_1 l}{18EA_1}$$

10-7 Cont.

$$\omega_1^2 = \left(\frac{6}{26} \times 18\right) \left(\frac{E}{\rho l^2}\right) = 4.1538 \left(\frac{E}{\rho l^2}\right)$$

$$\omega_1 = 2.038 \sqrt{\frac{EA_1}{m_1 l}} = \underline{\underline{2.038 \sqrt{\frac{E}{\rho l^2}}}} \quad \begin{array}{l} \text{for 1-element} \\ \text{solution} \\ \text{for Longit. vib.} \end{array}$$

where A_1 = cross section area at ①

$M_1 = m_1 l$ = mass of entire uniform cylinder of radius R_1 and length l .

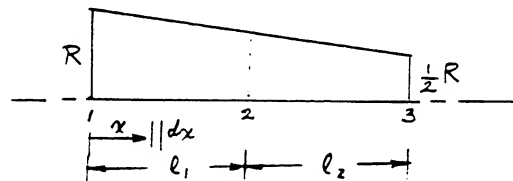
If we have a uniform cylindrical shell of Radius R_1 and length l , with one end fixed with other end free, treated like a helical spring with effective mass $\frac{m_1}{3}$ at free end, the natural frequency will be

$$\omega_1 = \sqrt{\frac{EA_1}{(\frac{m_1}{3})l}} = 1.732 \sqrt{\frac{EA_1}{m_1 l}}$$

10-8 Same conical tube as in 10-7 treated as two elements of equal length, 1-2 and 2-3.

$$r_{1-2} = R \left(1 - \frac{1}{4} \frac{x}{l_1}\right)$$

$$\xi = \frac{x}{l_1}$$



$$m_{ij} = \int \varphi_i \varphi_j dm = \int \varphi_i \varphi_j 2\pi \rho t r_{1-2} dx = 2\pi \rho t R \int_0^1 \left(1 - \frac{1}{4}\xi\right) \varphi_i \varphi_j l_1 d\xi$$

$$m_{11} = 2\pi \rho t R l_1 \int_0^1 \left(1 - \frac{1}{4}\xi\right) (1-\xi)^2 d\xi = 2\pi \rho t R l_1 (.3125)$$

$$m_{12} = \dots \int_0^1 \left(1 - \frac{1}{4}\xi\right) (1-\xi) \xi d\xi = \dots (-.1458)$$

$$m_{22} = \dots \int_0^1 \left(1 - \frac{1}{4}\xi\right) \xi^2 d\xi = \dots (.2708)$$

10-8 Cont.

$$m = 2\pi \rho t R l_1 \begin{bmatrix} .3125 & .1458 \\ .1458 & .2708 \end{bmatrix}$$

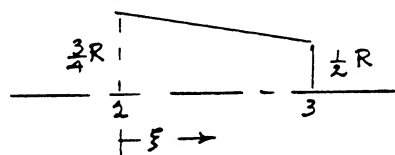
$$k_{ij} = \int_0^l EA \varphi_i' \varphi_j' d\xi \quad \therefore k_{11} = \int_0^l 2\pi R E \left(1 - \frac{1}{4}\xi\right) \left(-\frac{1}{4}\right)^2 d\xi$$

$$= \frac{2\pi R E}{l_1} (.8750)$$

$$k_{12} = -k_{11}, \quad k_{22} = k_{11}$$

$$k = \frac{2\pi R E}{l_1} \begin{bmatrix} .8750 & -.8750 \\ -.8750 & .8750 \end{bmatrix}$$

Element 2-3



$$r = \frac{3}{4} R \left(1 - \frac{1}{3}\xi\right)$$

$$A = 2\pi t r = 2\pi t R \frac{3}{4} \left(1 - \frac{1}{3}\xi\right)$$

$$dm = 2\pi \rho t r dx = 2\pi \rho t R \left(1 - \frac{1}{3}\xi\right) l_1 d\xi$$

$$m_{11} = \frac{3}{2} \pi \rho t R l_1 \int_0^1 \left(1 - \frac{1}{3}\xi\right) \left(1 - \xi\right)^2 d\xi = \frac{3}{2} \pi \rho t R l_1 (.3056)$$

$$m_{12} = \frac{3}{2} \pi \rho t R l_1 (.1389), \quad m_{22} = \frac{3}{2} \pi \rho t R l_1 (.2500)$$

$$m = \frac{3}{2} \pi \rho t R l_1 \begin{bmatrix} .3056 & .1389 \\ .1389 & .2500 \end{bmatrix} = 2\pi \rho t R l_1 \begin{bmatrix} .2292 & .1042 \\ .1042 & .1875 \end{bmatrix}$$

$$k_{11} = \frac{3}{2} \pi \rho t \frac{R}{l_1} \int_0^1 \left(1 - \frac{1}{3}\xi\right) d\xi = \frac{3}{2} \pi \rho t \frac{R}{l_1} \left(\frac{5}{6}\right)$$

$$k_{12} = -k_{11}, \quad k_{22} = k_{11}$$

$$\therefore k = \frac{3}{2} \pi \times \frac{5}{6} \rho t \frac{R}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

10-8 Cont. Assemble element matrices

$$m = 2\pi \rho t R l_1 \begin{bmatrix} .3125 & .1458 & 0 \\ .1458 & (.2708 + 2292) & .1042 \\ 0 & .1042 & .1875 \end{bmatrix}$$

$$k = \frac{\pi R E t}{l_1} \begin{bmatrix} 1.75 & -1.75 \\ -1.75 & (1.75 + 1.25) & -1.25 \\ & -1.25 & 1.25 \end{bmatrix} \quad \lambda = \frac{\omega^2 \rho l_1}{E}$$

With $u_1 = 0$, eqs. of motion leads to chara. eq.

$$\left| -\lambda \begin{bmatrix} .5000 & .1042 \\ .1042 & .1875 \end{bmatrix} + \begin{bmatrix} 3.00 & -1.25 \\ -1.25 & 1.25 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (3.00 - .500\lambda) & -(1.25 + .1042\lambda) \\ -(1.25 + .1042\lambda) & (1.25 - .1875\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 17.46\lambda + 26.380 = 0$$

$$\lambda_1 = 1.6708$$

$$\omega_1 = 1.2926 \sqrt{\frac{E}{\rho l_1^2}}$$

$$\lambda_2 = 15.7892$$

$$\omega_2 = 3.9736 \sqrt{\frac{E}{\rho l_1^2}}$$

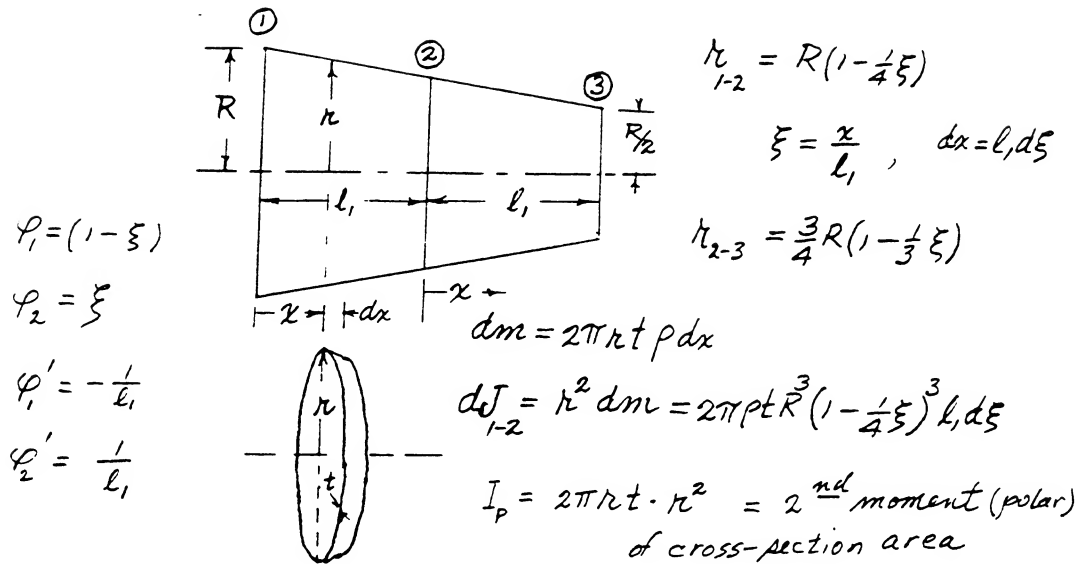
$$l_1 = \frac{l}{2}$$

$$\therefore \underline{\underline{\omega_1 = 2.585 \sqrt{\frac{E}{\rho l^2}}}}$$

$$\underline{\underline{\omega_2 = 7.947 \sqrt{\frac{E}{\rho l^2}}}}$$

} longitudinal vibr (2 elements)

10 - 9(a) Torsional Vibration of Conical Shell.



Stiffness of element 1-2.

$$\begin{aligned}
 k_{ij} &= G \int_0^{l_1} I_p \phi_i' \phi_j' dx = G \cdot 2\pi t R^3 \int_0^1 (1 - \frac{1}{4}\xi)^3 \phi_i' \phi_j' l_1 d\xi \\
 k_{11} &= G \cdot 2\pi t R^3 \int_0^1 (1 - \frac{1}{4}\xi)^3 (\frac{1}{l_1}) l_1 d\xi = \frac{G 2\pi t R^3}{l_1} \int_0^1 (1 - \frac{3}{4}\xi + \frac{3}{16}\xi^2 - \frac{1}{64}\xi^3) d\xi \\
 &= \frac{G 2\pi t R^3}{l_1} (1 - \frac{3}{8} + \frac{1}{16} - \frac{1}{256}) = \frac{G 2\pi t R^3}{l_1} (.6836) \\
 k_{12} &= k_{21} = -k_{11}, \quad k_{22} = k_{11}
 \end{aligned}$$

$$\therefore K_{1-2} = \frac{G 2\pi t R^3}{l_1} (.6836) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness of element 2-3

$$\begin{aligned}
 k_{11} &= \frac{G 2\pi t (.75R)^3}{l_1} \int_0^1 (1 - \frac{1}{3}\xi)^3 d\xi = \frac{G 2\pi t R^3}{l_1} (.4219) \int_0^1 (1 - \xi + \frac{1}{3}\xi^2 - \frac{1}{27}\xi^3) d\xi \\
 &= \frac{G 2\pi t R^3}{l_1} (.4219) (.6019) = \frac{G 2\pi t R^3}{l_1} (.2539) \\
 K_{2-3} &= \frac{G 2\pi t R^3}{l_1} (.2539) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{G 2\pi t R^3}{l_1} (.6836) \begin{bmatrix} .3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix}
 \end{aligned}$$

10-9 (a) Cont.

$$\text{Combined stiffness} = \frac{G 2 \pi t R^3}{l/2} (.6836) \begin{bmatrix} 1 & -1 & \\ -1 & 1.3714 & -.3714 \\ & -.3714 & .3714 \end{bmatrix}$$

For $\theta_1 = 0$

$$K\theta = (1.3672) \frac{G 2 \pi t R^3}{L} \begin{bmatrix} 1.3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

Mass matrix, element 1-2

$$J_{ij} = \int_0^1 \varphi_i \varphi_j dJ = 2 \pi \rho t R^3 \int_0^1 \left(1 - \frac{1}{4} \xi\right)^3 \varphi_i \varphi_j l_1 d\xi$$

$$\begin{aligned} J_{11} &= 2 \pi \rho t R^3 \int_0^1 \left(1 - \frac{1}{4} \xi\right)^3 (1 - \xi)^2 l_1 d\xi \\ &= 2 \pi \rho t R^3 l_1 \int_0^1 \left(1 - \frac{11}{4} \xi + \frac{43}{16} \xi^2 - \frac{73}{64} \xi^3 + \frac{7}{32} \xi^4 - \frac{1}{64} \xi^5\right) d\xi \\ &= \quad \quad \left(1 - \frac{11}{8} + \frac{43}{48} - \frac{73}{256} + \frac{7}{160} - \frac{1}{384}\right) \\ &= 2 \pi \rho t R^3 l_1 (.2768) \end{aligned}$$

$$\begin{aligned} J_{12} &= 2 \pi \rho t R^3 l_1 \int_0^1 \left(1 - \frac{1}{4} \xi\right)^3 (1 - \xi) \xi d\xi \\ &\quad \quad \quad \int_0^1 \left(\xi - \frac{7}{4} \xi^2 + \frac{15}{16} \xi^3 - \frac{13}{64} \xi^4 + \frac{1}{64} \xi^5\right) d\xi \\ &= 2 \pi \rho t R^3 l_1 (.1130) \end{aligned}$$

$$\begin{aligned} J_{22} &= 2 \pi \rho t R^3 l_1 \int_0^1 \left(1 - \frac{1}{4} \xi\right)^3 \xi^2 d\xi = \int_0^1 \left(\xi^2 - \frac{3}{4} \xi^3 + \frac{3}{16} \xi^4 - \frac{1}{64} \xi^5\right) d\xi () \\ &= 2 \pi \rho t R^3 l_1 \left(\frac{1}{3} - \frac{3}{16} + \frac{3}{80} - \frac{1}{384}\right) = 2 \pi \rho t R^3 l_1 (.1804) \end{aligned}$$

10-9(a) Cont.

$$J_{1-2} = 2\pi\rho t R^3 l_1 \begin{bmatrix} .2768 & .1130 \\ .1130 & .1804 \end{bmatrix} \quad (l_1 = \frac{l}{2})$$

Mass matrix, element 2-3 $h_{2-3} = \frac{3}{4} R (1 - \frac{1}{3} \xi)$

$$dJ_{2-3} = 2\pi h t \rho \cdot h^2 = 2\pi\rho t \left(\frac{3}{4} R\right)^3 \left(1 - \frac{1}{3} \xi\right)^3$$

$$J_{22} = 2\pi\rho t R^3 (.4219) \int_0^1 \left(1 - \frac{1}{3} \xi\right)^3 (1 - \xi)^2 l_1 d\xi$$

$$\therefore l_1 \int_0^1 \left(1 - 3\xi + \frac{10}{3} \xi^2 - 1.7036 \xi^3 + .40737 \xi^4 - .03704 \xi^5\right) d\xi$$

$$= 2\pi\rho t R^3 l_1 (.4219) (.2605) = 2\pi\rho t R^3 l_1 (.1099)$$

$$J_{23} = \text{same term} \times l_1 \int_0^1 \left(1 - \frac{1}{3} \xi\right)^3 (1 - \xi) \xi d\xi$$

$$= \therefore \int_0^1 \left(\xi - 2\xi^2 + \frac{4}{3} \xi^3 - \frac{10}{27} \xi^4 + \frac{1}{27} \xi^5\right) d\xi$$

$$= 2\pi\rho t R^3 l_1 (.4219) (.0988) = 2\pi\rho t R^3 l_1 (.0417)$$

$$J_{33} = \text{same term} \times l_1 \int_0^1 \left(1 - \frac{1}{3} \xi\right)^3 \xi^2 d\xi = 2\pi\rho t R^3 l_1 (.4219) (.1438)$$

$$= 2\pi\rho t R^3 l_1 (.0607)$$

Combined mass term with $\theta_1 = 0$

$$2\pi\rho t R^3 l_1 \begin{bmatrix} (.1099 + .1804) & .0417 \\ .0417 & .0607 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} \quad (l_1 = \frac{l}{2})$$

$$J = \pi\rho t R^3 l \begin{bmatrix} .2903 & .0417 \\ .0417 & .0607 \end{bmatrix}$$

10-9(a) Cont.

Eq. of motion: Let $\lambda = \frac{\omega^2 \rho l^2}{1.3672 G 2\pi R/l}$

$$\lambda = \frac{\omega^2 \rho l^2}{2.734 G}$$

$$\left[-\lambda \begin{bmatrix} .2903 & .0417 \\ .0417 & .0607 \end{bmatrix} + \begin{bmatrix} 1.3714 & -.3714 \\ -.3714 & .3714 \end{bmatrix} \right] \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} (1.3714 - .2903\lambda) & -(.3714 + .0417\lambda) \\ -(.3714 - .0417\lambda) & (.3714 - .0607\lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 10.069\lambda + 23.358 = 0$$

$$\lambda_1 = 3.624$$

$$\lambda_2 = 6.438$$

$$\omega_1^2 = 9.908 \frac{G}{\rho l^2}$$

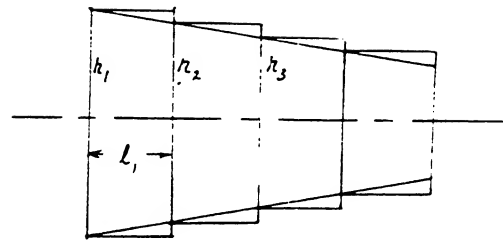
$$\omega_2^2 = 17.603 \frac{G}{\rho l^2}$$

$$\underline{\underline{\omega_1 = 3.147 \sqrt{\frac{G}{\rho l^2}}}}$$

$$\underline{\underline{\omega_2 = 4.196 \sqrt{\frac{G}{\rho l^2}}}}$$

Results unchecked.

10-9(b)



Torsion problem.

Assume short elements \therefore constant radius for each section.

Element stiffness:

$$A_i = 2\pi r_i t, \quad I_{p_i} = A_i r_i^2, \quad G I_{p_i} = G A_i r_i^2$$

$$K_i = \frac{G I_{p_i}}{l_i} = G \frac{2\pi t r_i^3}{l_i}$$

Assume straight line variation in twist $\therefore \varphi_{1_i}, \varphi_{2_i}$ applies to each element $\varphi_{1_i} = (1 - \xi), \quad \varphi_{2_i} = \xi$

where $\xi = \frac{x}{l_i}$. Then $K_i = \frac{G I_{p_i}}{l_i} \int_0^1 \varphi_{1_i}' \varphi_{2_i}' l_i d\xi$

$$\int_0^1 \varphi_{1_i}' \varphi_{2_i}' l_i d\xi = \pm 1$$

$$K \text{ for element} = G \left(\frac{2\pi t r_i^3}{l_i} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad l_i = \frac{l}{n}$$

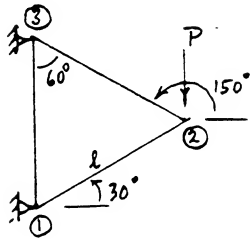
Element mass:

$$J \text{ for element} = \frac{2\pi t \rho l_i r_i^3}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

N section eq.

$$\rho \frac{2\pi t l}{6n} \begin{bmatrix} 2r_1^3 & r_1^3 & & \\ r_1^3 & 2(r_1^3 + r_2^3) & r_2^3 & \\ & r_2^3 & 2(r_2^3 + r_3^3) & r_3^3 \\ & & r_3^3 & 2(r_3^3 + r_4^3) \\ & & & \ddots \\ & & & & \text{etc.} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \vdots \end{Bmatrix} + \frac{G 2\pi t n}{l} \begin{bmatrix} r_1^3 & -r_1^3 & & \\ -r_1^3 & (r_1^3 + r_2^3) & -r_2^3 & \\ & -r_2^3 & (r_2^3 + r_3^3) & \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \end{Bmatrix}$$

10-10



Element 1-2 $\alpha = 30^\circ$ $C = .866$, $S = .500$
 $C^2 = .750$ $S^2 = .250$ $CS = .433$

$$\bar{k}_{12} = \frac{EA}{l} \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & .750 & .433 \\ \dots & \dots & .433 & .250 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

Element 2-3, $\alpha = 150^\circ$ $C = -.866$ $S = .50$

$$\bar{k}_{23} = \frac{EA}{l} \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & .750 & .433 \\ \dots & \dots & .433 & .250 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \end{Bmatrix}$$

Assemble

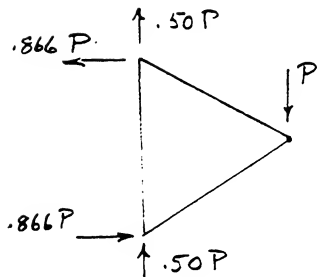
$$\begin{Bmatrix} F_{2x} = 0 \\ F_{2y} = -P \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \frac{EA}{l} \begin{bmatrix} 1.50 & 0 & \dots & \dots \\ 0 & .50 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 = 0 \\ \bar{v}_3 = 0 \end{Bmatrix}$$

$$F_{2x} = 0 = \frac{EA}{l} 1.50 \bar{u}_2 \quad \therefore \bar{u}_2 = 0$$

$$F_{2y} = -P = \frac{EA}{l} .50 \bar{v}_2 \quad \therefore \bar{v}_2 = -\frac{2Pl}{EA}$$

$$F_{3x} = \frac{EA}{l} .433 \bar{v}_2 = \frac{EA}{l} .433 \left(-\frac{2Pl}{EA} \right) = -.866P$$

$$F_{3y} = -.25 \bar{v}_2 \frac{EA}{l} = -.25 \frac{EA}{l} \left(-\frac{2Pl}{EA} \right) = .50P$$



10-11 See Example 10.4-1 for the 6×6 general matrix equation which applies to this problem. With zero displacements for u_1 , v_2 , u_3 and v_3 the equation reduces to

$$\begin{Bmatrix} F_{1x} \\ F_{1y} = 0 \\ F_{2x} = P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \left(\frac{EA}{25L} \right) \begin{bmatrix} 12 & -16 \\ 50.67 & -12 \\ -12 & 47.25 \\ -9 & 12 \\ 0 & -31.25 \\ -41.67 & 0 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 \\ u_2 \\ v_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

columns crossed out

The displacements v_1 and u_2 are available from the second and third equations

$$0 = 50.67 v_1 - 12 u_2 \quad \therefore v_1 = .2368 u_2$$

$$P = \left(\frac{EA}{25L} \right) [-12 v_1 + 47.25 u_2] = \left(\frac{EA}{25L} \right) 44.41 u_2$$

$$\therefore u_2 = \frac{PL}{EA} (.5629), \quad v_1 = (.1333) \frac{PL}{EA}$$

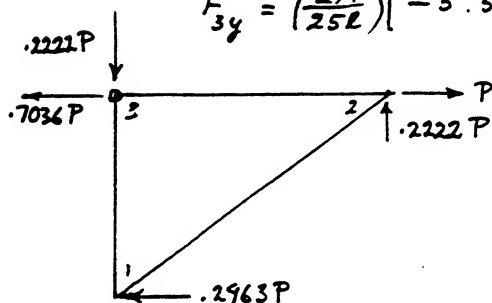
The reaction forces are then available from the 4 remaining equations:

$$F_{1x} = \left(\frac{EA}{25L} \right) [1.5996 - 9.0064] \frac{PL}{EA} = -.2963 P$$

$$F_{2y} = \left(\frac{EA}{25L} \right) [-1.1997 + 6.7548] \frac{PL}{EA} = 0.2222 P$$

$$F_{3x} = \left(\frac{EA}{25L} \right) [-17.5906] \frac{PL}{EA} = -.7036 P$$

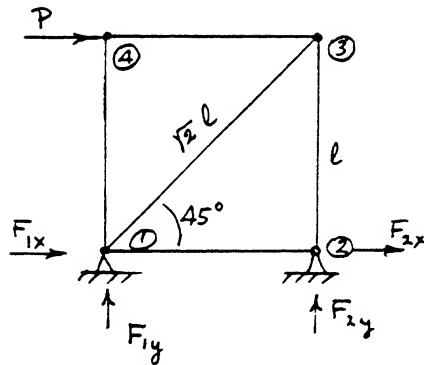
$$F_{3y} = \left(\frac{EA}{25L} \right) [-5.5746] \frac{PL}{EA} = -.2222 P$$



$$\text{Check } \sum M_3 = 0$$

$$\begin{aligned} \sum F_x &= P - F_{1x} - F_{3x} \\ &= P - .2963 P - .7036 P = 0 \end{aligned}$$

10-12



$$\bar{k}_{12} = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \bar{m}_{12} = \frac{ml}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Element 1-2, $\alpha=0$, $s=0$, $c=1$

$$\bar{k}_{1-2} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Element 2-3 $\alpha=90^\circ$, $s=1$, $c=0$

$$\bar{k}_{2-3} = \frac{EA}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Element 3-4 $\alpha=180^\circ$, $s=0$, $c=-1$

$$\bar{k}_{3-4} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

Element 4-1 $\alpha=270^\circ$, $s=-1$, $c=0$

$$\bar{k}_{4-1} = \frac{EA}{l} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} 4 \\ 1 \end{matrix}$$

Element 1-3 $\alpha=45^\circ$, $s = \frac{\sqrt{2}}{2}$, $c = \frac{\sqrt{2}}{2}$ Length = $\sqrt{2}l$

$$\bar{k}_{13} = \frac{EA}{\sqrt{2}l} \begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & -.5 & -.5 \\ -.5 & -.5 & .5 & .5 \\ -.5 & -.5 & .5 & .5 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix}$$

10-12 Cont.

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ \bar{F}_{2x} \\ \bar{F}_{2y} \\ \bar{F}_{3x} \\ \bar{F}_{3y} \\ \bar{F}_{4x} \\ \bar{F}_{4y} \end{Bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

Since all displacements for ① and ② are zero, we need only the lower part of this matrix or the lower right section of K

However for instructional purposes we will fill in the elements into the 8×8 space for K .

There is no problem with placement of \bar{K}_{12} , \bar{K}_{23} and \bar{K}_{34} . Element \bar{K}_{41} and \bar{K}_{43} must be placed as follows:

Element 4-1

1	0	0		0	0	1
	0	1		0	-1	
2			\bar{K}_{21}			2
				\bar{K}_{23}		
3					\bar{K}_{34}	3
	0	0		0	0	
4	0	-1		0	1	4

dotted squares indicate placement of \bar{K}_{12} , \bar{K}_{23} , \bar{K}_{34}

Element 1-3

1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$		$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$		$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	
2						2
3	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	3
	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
4						4

superimpose this onto diagram on left.

Considering only those terms in lower right quarter, we have

$$\begin{Bmatrix} \bar{F}_{3x} = 0 \\ \bar{F}_{3y} = 0 \\ \bar{F}_{4x} = P \\ \bar{F}_{4y} = 0 \end{Bmatrix} = \frac{EA}{l} \begin{bmatrix} (1 + \frac{\sqrt{2}}{2}) & \frac{\sqrt{2}}{2} & -1 & 0 \\ \frac{\sqrt{2}}{2} & (1 + \frac{\sqrt{2}}{2}) & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

10-12 Cont. The last matrix give

$$(1) \quad 0 = \frac{EA}{l} \left[(1 + \frac{5}{12}) \bar{u}_3 + \frac{5}{12} \bar{v}_3 - \bar{u}_4 \right]$$

$$(2) \quad 0 = \frac{EA}{l} \left[\frac{5}{12} \bar{u}_3 + (1 + \frac{5}{12}) \bar{v}_3 \right] \rightarrow \bar{u}_3 = -3.828 \bar{v}_3$$

$$(3) \quad P = \frac{EA}{l} \left[-\bar{u}_3 + \bar{u}_4 \right]$$

$$(4) \quad 0 = \frac{EA}{l} \left[\bar{v}_4 \right] \therefore \bar{v}_4 = 0$$

add (1) and (3) to eliminate \bar{u}_4 + subst (2) into it.

$$\frac{Pl}{EA} = \frac{5}{12} \bar{u}_3 + \frac{5}{12} \bar{v}_3 \rightarrow \bar{u}_3 + \bar{v}_3 = \frac{1}{.3536} \frac{Pl}{AE}$$

$$\bar{u}_3 - \frac{1}{3.828} \bar{u}_3 = \frac{Pl}{.3536 AE} \therefore \bar{u}_3 = 3.828 \frac{Pl}{AE}$$

$$\bar{v}_3 = -\frac{Pl}{AE}$$

$$\bar{u}_4 = 4.828 \frac{Pl}{AE}$$

Complete Equation

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \frac{EA}{l} \begin{bmatrix} (1+\frac{5}{12}) & \frac{5}{12} & -1 & 0 & -\frac{5}{12} & -\frac{5}{12} & 0 & 0 \\ \frac{5}{12} & (1+\frac{5}{12}) & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -\frac{5}{12} & -\frac{5}{12} & 0 & 0 & (1+\frac{5}{12}) & \frac{5}{12} & -1 & 0 \\ -\frac{5}{12} & -\frac{5}{12} & 0 & -1 & \frac{5}{12} & (1+\frac{5}{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

Mass matrix

$$\bar{m} = \frac{ml}{6} \begin{bmatrix} 2c^2 & 2cs & c^2 & -cs \\ 2cs & 2s^2 & cs & s^2 \\ c^2 & cs & 2c^2 & 2cs \\ cs & s^2 & 2cs & 2s^2 \end{bmatrix}$$

Element a: $\alpha = 0$ $\cos(\alpha) = 1$, $\sin(\alpha) = 0$

$$= \frac{ml}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

Element b: $\alpha = 90$ $\cos(\alpha) = 0$, $\sin(\alpha) = 1$

$$= \frac{ml}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \end{Bmatrix}$$

element D: $\alpha = 180$ $\cos(\alpha) = -1$ $\sin(\alpha) = 0$

$$= \frac{ml}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

Element E: $\alpha = 270$ $\sin(270) = -1$ $\cos(270) = 0$

$$= \frac{ml}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_4 \\ \bar{v}_4 \\ \bar{u}_1 \\ \bar{v}_1 \end{Bmatrix}$$

$$c^2 = \frac{1}{2} = cs = s^2$$

Element C: $\alpha = 225$ $\cos(225) = -1/\sqrt{2}$ $\sin(\alpha) = -1/\sqrt{2}$

$$= \frac{ml}{6} \begin{bmatrix} 1/\sqrt{2} & 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_1 \\ \bar{v}_1 \end{Bmatrix}$$

ADD Elements a+b

$$\frac{ml}{6} \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \end{Bmatrix}$$

Elements a, b + D

$$\frac{ml}{b} \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

rewrite element E as

$$\frac{ml}{b} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

elements a, b, D + E

$$\frac{ml}{b} \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

rewrite element C as:

$$\frac{ml}{6} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

elements a, b, D, E & C

$$\frac{ml}{6} \begin{bmatrix} 2+\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 2+\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 2+\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 2+\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

→ Now impose zero displacement @ 1 & 2

$$\rightarrow \frac{ml}{6} \begin{bmatrix} 2+\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 & 0 \\ \frac{\sqrt{2}}{2} & 2+\frac{\sqrt{2}}{2} & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

10-12 cont

=

$$\rightarrow \begin{cases} F_{3x} = 0 \\ F_{3y} = 0 \\ F_{4x} = P \\ F_{4y} = 0 \end{cases} = \frac{EA}{L} \begin{bmatrix} 1 + \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & -1 & 0 \\ \frac{5}{\sqrt{2}} & 1 + \frac{5}{\sqrt{2}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix} + \frac{mL}{6} \begin{bmatrix} 2 + \sqrt{2} & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 2 + \sqrt{2} & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{u}_4 \\ \bar{v}_4 \end{Bmatrix}$$

$$Q = \text{chol}(M)$$

$$a = Q^T * Q^{-1}$$

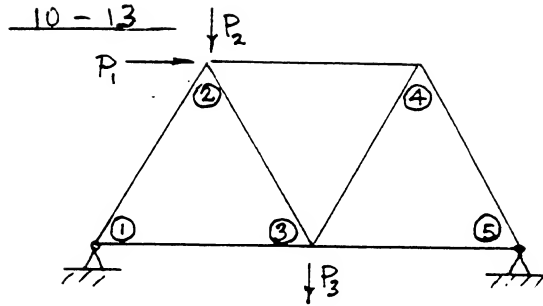
a =

$$\begin{bmatrix} 0.3964 & -0.0666 & -0.6018 & 0 \\ -0.0666 & 0.4571 & 0.3078 & 0 \\ -0.6018 & 0.3078 & 1.1350 & 0 \\ 0 & 0 & 0 & 0.5000 \end{bmatrix}$$

EDU> eig(a)

ans =

$$\begin{bmatrix} 0.3978 \\ 0.0360 \\ 1.5548 \\ 0.5000 \end{bmatrix}$$



Elements 1-2, 3-4, $\alpha = 60^\circ$ $C = .5$, $S = .866$, $CS = .433$
 $C^2 = .250$ $S^2 = .750$

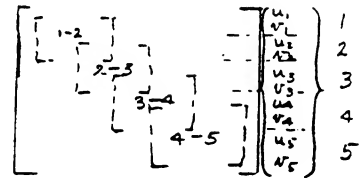
$$\bar{k} = \frac{EA}{l} \begin{bmatrix} .250 & .433 & -0.250 & -.433 \\ .433 & .750 & -.433 & -.750 \\ -0.250 & -.433 & .250 & .433 \\ .750 & .750 & .433 & .750 \end{bmatrix}$$

Elements 2-3, 4-5, $\alpha = 300^\circ$ $C = .5$ $S = -.866$ $CS = -.433$
 $C^2 = .250$ $S^2 = .750$

$$\bar{k} = \frac{EA}{l} \begin{bmatrix} .250 & -.433 & -.250 & .433 \\ & .750 & .433 & -.750 \\ & & .250 & -.433 \\ & & .750 & .750 \end{bmatrix}$$

Element 1-3, 2-4, 3-5, $\alpha = 0$ $C = 1$ $S = 0$ $CS = 0$
 $C^2 = 1$ $S^2 = 0$

$$\bar{k} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



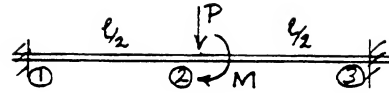
$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2-4

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Element 3-5

10-14



From Example 10.5-1 and noting that the ends 1 and 3 are fixed, the stiffness matrix and displacement vectors are:

$$\begin{Bmatrix} F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \frac{8EI}{l^3} \begin{bmatrix} 24 & 0 & -12 & 3l \\ 0 & 2l^2 & -3l & .5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & .5l^2 & -3l & l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ -M \end{Bmatrix} = \frac{8EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 2l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \bar{u}_2 = -\frac{Pl^3}{192EI} \quad \theta_2 = -\frac{Ml}{16EI}$$

$$M_3 = \frac{8EI}{l^3} [3l \bar{u}_2 + .5l^2 \theta_2] = \frac{8EI}{l^3} \left[3l \left(-\frac{Pl^3}{192EI} \right) + .5l^2 \left(-\frac{Ml}{16EI} \right) \right]$$

10-15 Refer to Example 10.5-1 with $u_3 = \theta_3 = 0$

$$\left[-\frac{\omega^2 ml}{840} \begin{bmatrix} 312 & 0 \\ 0 & 2l^2 \end{bmatrix} + \frac{8EI}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 2l^2 \end{bmatrix} \right] \begin{Bmatrix} u_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \left(\frac{\omega^2 ml}{840} \right) \left(\frac{l^3}{8EI} \right) = \frac{ml^4 \omega^2}{6720EI}$$

$$\begin{vmatrix} (24 - 312\lambda) & 0 \\ 0 & 2l^2(1 - \lambda) \end{vmatrix} = 0 \quad \lambda_1 = \frac{24}{312} = \frac{ml^4 \omega_1^2}{6720EI}$$

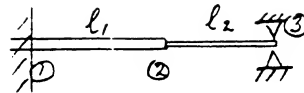
$$\lambda_2 = 1 = \frac{ml^4 \omega_2^2}{6720EI}$$

$$\therefore \omega_1 = 22.74 \sqrt{\frac{EI}{ml^4}} \rightarrow 1.6\% \text{ high}$$

$$\omega_2 = 81.98 \sqrt{\frac{EI}{ml^4}} \rightarrow 33\% \text{ high.}$$

$$\text{Exact values } \begin{cases} \omega_1 = 22.37 \sqrt{\frac{EI}{ml^4}} \\ \omega_2 = 61.67 \sqrt{\frac{EI}{ml^4}} \end{cases}$$

10-16



Element 1-2

$$\bar{k} = \frac{EI_1}{l_1^3} \begin{bmatrix} 12 & 6l_1 & 0 \\ 0 & 12 & -6l_1 \\ 0 & -6l_1 & 4l_1^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} \quad \bar{m} = \frac{m_1 l_1}{420} \begin{bmatrix} 156 & -22l_1 \\ -22l_1 & 4l_1^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{\theta}_2 \end{Bmatrix}$$

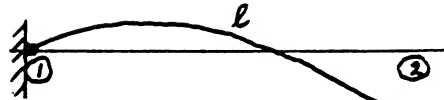
Element 2-3

$$\bar{k} = \frac{EI_2}{l_2^3} \begin{bmatrix} 12 & 6l_2 & 0 & -12 & 6l_2 \\ 0 & 12 & -6l_2 & 0 & -6l_2 \\ 0 & 6l_2 & 4l_2^2 & 0 & 2l_2^2 \\ -12 & -6l_2 & 0 & 12 & -6l_2 \\ 0 & -6l_2 & 2l_2^2 & 0 & 4l_2^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \\ \bar{u}_3 \\ \bar{\theta}_3 \end{Bmatrix} \quad \bar{m} = \frac{m_2 l_2}{420} \begin{bmatrix} 156 & 22l_2 & 0 & 54 & -13l_2 \\ 22l_2 & 4l_2^2 & 0 & 13l_2 & -3l_2^2 \\ 0 & 0 & 0 & 0 & 0 \\ 54 & 13l_2 & 0 & 156 & 22l_2 \\ -13l_2 & -3l_2^2 & 0 & -22l_2 & 4l_2^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{\theta}_3 \end{Bmatrix}$$

Free vibr. eq.

$$\frac{1}{420} \begin{bmatrix} 156(m_1 l_1 + m_2 l_2) & 22(m_2 l_2^2 - m_1 l_1^2) & -13 m_2 l_2^2 \\ 22(m_2 l_2^2 - m_1 l_1^2) & 4(m_2 l_2^3 + m_1 l_1^3) & -3 m_2 l_2^3 \\ -13 m_2 l_2^3 & -3 m_2 l_2^3 & 4 m_2 l_2^3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + E \begin{bmatrix} 12(\frac{I_2}{l_2^3} + \frac{I_1}{l_1^3}) & 6(\frac{I_2}{l_2^2} - \frac{I_1}{l_1^2}) & 6 \frac{I_2}{l_2^2} \\ 6(\frac{I_2}{l_2^2} - \frac{I_1}{l_1^2}) & 4(\frac{I_2}{l_2} + \frac{I_1}{l_1}) & 2 \frac{I_2}{l_2} \\ 6 \frac{I_2}{l_2^2} & 2 \frac{I_2}{l_2} & 4 \frac{I_2}{l_2} \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{\theta}_2 \\ \bar{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

10-17



From Eq. 10.2-1 and 10.2-10 the mass and stiffness matrices with associated displacement vector are:

$$M = \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ -22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \quad \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$K = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

For the pinned-free beam $v_1 = 0$ which eliminates the first column, and for the eigen problem, F_{ix} is of no interest so that the first row can also be deleted. Also when the characteristic equation $|\omega^2 M + K| = 0$ is solved, the l in the matrices factor out, and we can let $l = 1.0$ and rewrite the equation as:

$$\left| -\left(\frac{\omega^2 ml}{420}\right) \begin{bmatrix} 4 & 13 & -3 \\ 13 & 156 & -22 \\ -3 & -22 & 4 \end{bmatrix} + \left(\frac{EI}{l^3}\right) \begin{bmatrix} 4 & -6 & 2 \\ -6 & 12 & -6 \\ 2 & -6 & 4 \end{bmatrix} \right| = 0$$

It should be noted that the stiffness matrix has a determinant which is zero, and therefore cannot be inverted or decomposed. It is therefore necessary to decompose the M matrix.

10-17 cont

```
>>m=[ 4 13 -3 ; 13 156 -22; -3 -22 4]
```

m =

4	13	-3
13	156	-22
-3	-22	4

```
>>k=[ 4 -6 2; -6 12 -6; 2 -6 4]
```

k =

4	-6	2
-6	12	-6
2	-6	4

```
>>q=chol(m)
```

q =

2.0000	6.5000	-1.5000
0	10.6654	-1.1486
0	0	0.6563

```
>>qin=inv(q)
```

qin =

0.5000	-0.3047	0.6094
0	0.0938	0.1641
0	0	1.5236

```
>>a=qin'*k*qin
```

a =

1.0000	-0.8907	2.2503
-0.8907	0.8198	-2.3868
2.2503	-2.3868	10.6088

10-17 cont.

```
>>[v,d]=eig(a)
```

```
v =
```

```
    0.5916    0.7760    0.2186
    0.8043   -0.5495   -0.2262
    0.0555   -0.3096    0.9492
```

```
d =
```

```
    1.0000    0.0000    0.0000
    0.0000    0.7329    0.0000
    0.0000    0.0000   11.6957
```

$$\lambda = \left(\frac{\omega^2 m l^4}{420 EI} \right)$$

$$\omega_1 = 0$$

$$\omega_2 = 17.54 \sqrt{\frac{EI}{m l^4}}$$

$$\omega_3 = 70.39 \sqrt{\frac{EI}{m l^4}}$$

```
>>x=qin*v
```

```
x =
```

```
    0.0845    0.3668    0.7567
    0.0845   -0.1023    0.1345
    0.0845   -0.4717    1.4463
```

exact values

$$\omega_0 = 0$$

$$\omega_1 = 15.4 \sqrt{\frac{EI}{m l^4}}$$

$$\omega_2 = 49.96 \sqrt{\frac{EI}{m l^4}}$$

```
>>
```

Conclusion: One element model is unsatisfactory

Alternative: use the program beam.m
see the next problem for details
about the use of this code.

10-18

1

»

What is the length of the bar? 1

len =

1

How many elements would you like? 2

n =

2

What is the mass of the beam ? 1

mass =

1

What is the elastic modulus of the beam ? 1

E =

1

What is the moment of inertia of the beam?1

I =

1

v =

0.1311	0.4801	-0.6584	0.5245	0.2092
-0.0266	-0.3339	0.3855	0.5244	0.6813
0.3884	0.6812	0.4446	-0.2905	0.3211
-0.2459	-0.1322	-0.4223	-0.5961	0.6233
0.8779	-0.4202	-0.2049	-0.1008	0.0219

10-18 cont

≡

lambda =

1.0e+04 *

6.2116	0	0	0	0
0	1.8376	0	0	0
0	0	0.3116	0	0
0	0	0	0.0241	0
0	0	0	0	0.0000

omega =

249.2306	0	0	0	0
0	135.5595	0	0	0
0	0	55.8228	0	0
0	0	0	15.5142	0
0	0	0	0	0.0000

Exact ^{49.96} 15.4 0

x =

22.2515	29.2961	-13.3294	5.4582	1.7321
0.4353	-1.1285	0.4848	1.1871	0.8660
28.3733	10.7305	11.8711	-2.7048	1.7321
3.3798	-2.2721	-2.0447	-2.0225	1.7321
69.3715	-33.2071	-16.1938	-7.9688	1.7321

10-19

Input

$l = 1$
 $n = 6$
 $mass = 1$
 $E = 1$
 $I = 1$

/0-19 cont

=

omega =

1.0e+03 *

Columns 1 through 4

2.1561	0	0	0
0	1.4025	0	0
0	0	1.7392	0
0	0	0	1.0621
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Columns 5 through 8

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0.7901	0	0	0
0	0.5804	0	0
0	0	0.4248	0
0	0	0	0.2782
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Columns 9 through 12

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

10-19 cont

0	0	0	0
0	0	0	0
0	0	0	0
0.1805	0	0	0
0	0.1048	0	0
0	0	0.0500	0
0	0	0	0.0154
0	0	0	0

exact 49.96 exact 15.4

Column 13

0
0
0
0
0
0
0
0
0
0
0
0
0
0
0 + 0.0000i

exact 0

x =

Columns 1 through 7

10.1121	-99.2702	-129.9937	-75.3367	57.9828	45.6155	37.4875
0.0505	0.8057	0.1529	1.2468	-1.2426	-0.8243	-0.1698
12.3264	-60.8796	-123.0672	-9.3374	-22.3957	-36.0458	-36.9790
0.1232	0.9882	0.2894	0.3091	0.9599	1.3027	0.3350
19.9391	24.5986	-103.0258	73.0222	-40.6822	11.3519	35.4673
0.2497	0.4064	0.3951	-1.1702	0.5011	-1.2346	-0.4910
36.2843	91.0508	-72.0052	27.4385	53.8227	18.1045	-32.9921
0.4865	-0.4901	0.4590	-0.5987	-1.3465	0.6475	0.6353
68.5425	87.0711	-33.3062	-66.2113	-0.8834	-39.9837	29.6521
0.9946	-1.0285	0.4871	1.0447	0.5653	0.1777	-0.7220
132.3452	15.2116	9.2843	-43.2898	-52.5323	44.3715	-24.7305
5.9261	-2.1565	1.3595	2.2411	2.2287	-2.1771	1.8257
358.7717	-103.5514	74.0764	89.3228	71.9851	-56.8220	39.1004

Columns 8 through 13

10-1900n t

≡

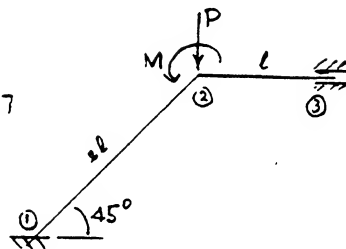
-23.3039	-19.2839	-14.5747	10.0327	5.4017	1.7321
-0.5933	-1.1557	-1.4162	1.3123	0.8337	0.2887
21.4342	11.7057	1.8988	3.8566	4.2193	1.7321
1.0915	1.4029	0.3698	1.0115	1.2992	0.5774
-16.1249	5.0703	14.0933	-7.0208	1.1312	1.7321
-1.4142	-0.5490	1.3266	-0.5131	1.1698	0.8660
8.2338	-17.8814	-5.4988	-9.0513	-2.6879	1.7321
1.5151	-0.7502	-0.6786	-1.3227	0.4390	1.1547
1.0618	16.4539	-12.2908	0.6700	-5.8784	1.7321
-1.2954	1.3320	-0.9506	-0.2342	-0.7036	1.4434
-8.9312	-3.7942	10.7315	11.4386	-7.5543	1.7321
2.0371	-2.0418	2.0180	2.0047	-2.0005	1.7321
34.3242	-27.4611	20.6364	14.1729	-7.8613	1.7321

10-20

Element 1-2 $\alpha = 45^\circ$ $C = .707$ $S = .707$

$$C^2 = S^2 = CS = .50$$

$$l_{12} = 2l$$



$$\bar{k} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} (.5R_{12} + 6) & .50(R_{12} - 12) & 12l \cdot .707 \\ .50(R_{12} - 12) & .5(R_{12} + 12) & -12l \cdot .707 \\ 12l \cdot .707 & -12l \cdot .707 & 16l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} \quad \text{All } R_{1-2} = \frac{A}{I} (2l)^2 = \frac{4Al^2}{I}$$

Element 2-3 $\alpha = 0$ $C = 1$ $S = 0$ $l_{23} = l$

$$\bar{k} = \left(\frac{EI}{l^3} \right) \begin{bmatrix} R_{23} & 0 & 0 \\ 0 & 12 & 6l \\ 0 & 6l & 4l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} \quad \text{All } R_{2-3} = \frac{Al^2}{I} = \frac{1}{4} R_{1-2}$$

Assembled with all $R_5 = R_{12} = R$

$$\bar{k} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} (.5R + 2R + 6) & .5(R - 12) & .707 \times 12l \\ .5(R + 12) & .5(R + 12) + 96 & -.707 \times 12l + 48l \\ 12 \times .707 l & -.707 \times 12l + 48l & 16l^2 + 32l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}$$

10-20 CONT

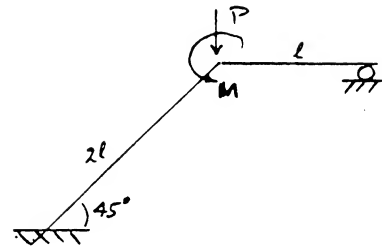
$$\begin{Bmatrix} 0 \\ -P \\ M \end{Bmatrix} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} (2.5R+6) & .5(R-12) & 8.484l \\ .5(R-12) & (.5R+102) & 39.52l \\ 8.484l & 39.52l & 48l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

$$R = \frac{4Al^2}{I}$$

10-21

Element 1-2

$$\bar{k} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} (.5R+6) & .5(R-12) & 8.484l \\ .5(R-12) & .5(R+12) & -8.484l \\ 8.484l & -8.484l & 16l^2 \end{bmatrix}$$



Element 2-3

$$\bar{k} = \left(\frac{EI}{l^3} \right) \begin{bmatrix} R_{23} & 0 & 0 \\ 0 & 3 & 3l \\ 0 & 3l & 3l^2 \end{bmatrix} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} 2R & 0 & 0 \\ 0 & 24 & 24l \\ 0 & 24l & 24l^2 \end{bmatrix}$$

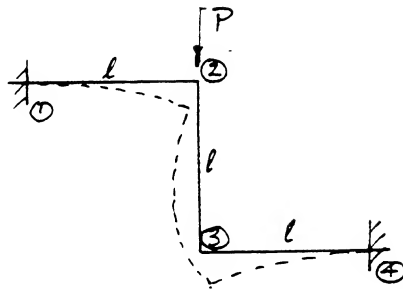
Assembled

$$\begin{Bmatrix} 0 \\ -P \\ M \end{Bmatrix} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} (2.5R+6) & .5(R-12) & 8.484l \\ .5(R-12) & (.5R+30) & 15.52l \\ 8.484l & 15.52l & 40l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

From 1st of above equation express \bar{u}_2 in terms of \bar{v}_2 and θ_2 and write

$$\begin{Bmatrix} -P \\ M \end{Bmatrix} = \left(\frac{EI}{8l^3} \right) \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

10-22a)



$$\begin{Bmatrix} \bar{F}_{1x} \\ \vdots \\ \bar{M}_4 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 \times 12 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \vdots \\ \bar{\theta}_4 \end{Bmatrix}$$

Elements 1-2 and 3-4 $\alpha=0$ $C=1$, $S=0$

$$\bar{k}_{1-2} = \bar{k}_{3-4} = \frac{EI}{l^3} \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix}$$

vectors $\begin{Bmatrix} \bar{u}_1=0 \\ \bar{v}_1=0 \\ \bar{\theta}_1=0 \\ \vdots \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}$ or $\begin{Bmatrix} \bar{u}_3 \\ \bar{v}_3 \\ \bar{\theta}_3 \\ \vdots \\ \bar{u}_4=0 \\ \bar{v}_4=0 \\ \bar{\theta}_4=0 \end{Bmatrix}$

Element 2-3 $\alpha=270^\circ$ or -90° $C=0$, $S=-1$

$$\bar{k}_{2-3} = \frac{EI}{l^3} \begin{bmatrix} 12 & 0 & 6l & -12 & 0 & 6l \\ 0 & R & 0 & 0 & -R & 0 \\ 6l & 0 & 4l^2 & -6l & 0 & 2l^2 \\ -12 & 0 & -6l & 12 & 0 & -6l \\ 0 & -R & 0 & 0 & R & 0 \\ 6l & 0 & 2l^2 & -6l & 0 & 4l^2 \end{bmatrix}$$

vector $\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \\ \vdots \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{\theta}_3 \end{Bmatrix}$

Only displacements involving ② and ③ need to be considered. Thus we need only to add lower right quarter of \bar{k}_{1-2} to upper left quarter of \bar{k}_{2-3} , and upper left qtr. of \bar{k}_{3-4} to lower right qtr. of \bar{k}_{2-3}

$$\begin{Bmatrix} \bar{F}_{2x}=0 \\ -P \\ \bar{M}_2=0 \\ \vdots \\ \bar{F}_{3x}=0 \\ \bar{F}_{3y}=0 \\ \bar{M}_3=0 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} (R+12) & 0 & 6l & -12 & 0 & 6l \\ 0 & (R+12) & -6l & 0 & -R & 0 \\ 6l & -6l & 8l^2 & -6l & 0 & 2l^2 \\ -12 & 0 & -6l & (2+12) & 0 & -6l \\ 0 & -R & 0 & 0 & (R+12) & 6l \\ 6l & 0 & 2l^2 & -6l & 6l & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \\ \vdots \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{\theta}_3 \end{Bmatrix}$$

symmetric matrix

10-22(a) Cont.

For no axial extension $\bar{u}_2 = -\bar{u}_3 = 0$

this eliminates 1st and 4th columns

Then $F_{2x} = F_{3x} = 0$ and the 1st and 4th rows

requires that $\theta_3 = -\theta_2$ which eliminates
the 1st and 4th rows, leaving the 4x4
matrix

$$\begin{Bmatrix} -P \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} (R+12) & -6l & -R & 0 \\ -6l & 8l^2 & 0 & 2l^2 \\ -R & 0 & (R+12) & 6l \\ 0 & 2l^2 & 6l & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix} \quad \text{symmetric}$$

Since $\bar{u}_3 = \bar{u}_2$ add 3rd column to 1st column.

Also since $\theta_3 = -\theta_2$, subtract 4th col. from 2nd col.

This also reduces the matrix to a 2x2

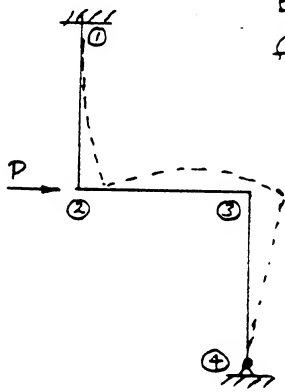
$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 6l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \theta_2 \end{Bmatrix}$$

Solving the two equations

$$l\theta_2 = \bar{u}_2$$

$$\boxed{\begin{aligned} \bar{u}_2 &= \frac{-Pl^3}{6EI} \\ \theta_2 &= \frac{-Pl^2}{6EI} \end{aligned}}$$

10-22(b)



Elements 1-2 and 2-3 are already available from Prob 10-22(a) by interchanging 1-2 and 2-3

The new element 3-4 is obtained:

with $\alpha = -90^\circ$ $C=0$, $S=-1$

Element 3-4

$$\frac{EI}{l^3} \begin{bmatrix} 3 & 0 & 3l & -3 & 0 & 3l \\ 0 & R & 0 & 0 & -R & 0 \\ 3l & 0 & 3l^2 & 0 & 0 & 0 \\ -3 & 0 & -3l & 3 & 0 & -3l \\ 0 & -R & 0 & 0 & R & 0 \\ 0 & 0 & 0 & -3l & 0 & 3l^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} -12 & 0 & 6l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & -R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6l & 0 & 2l^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (12+R) & 0 & -6l & -R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (12+R) & 6l & 0 & -12 & 6l & 0 & 0 & 0 & 0 & 0 & 0 \\ -6l & 6l & 8l^2 & 0 & -6l & 2l^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -R & 0 & 0 & (R+3) & 0 & 3l & -3 & 0 & 3l & 0 & 0 & 0 \\ 0 & -12 & -6l & 0 & (R+12) & -6l & 0 & -R & 0 & 0 & 0 & 0 \\ 0 & 6l & 2l^2 & 3l & -6l & 7l^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -3l & 3 & 0 & -3l & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3l & 0 & 3l^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \\ 0 \end{Bmatrix}$$

We need only the central part of the matrix

10-22(b) Cont.

$$\bar{u}_2 = \bar{u}_3 = 0 \quad \therefore \text{eliminate cols 2 \& 5}$$

$$\begin{Bmatrix} \bar{F}_{3x} = P \\ \bar{M}_2 \\ \bar{F}_{3x} \\ \bar{M}_3 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} (R+12) & -6l & -R & 0 \\ -6l & 8l^2 & 0 & 2l^2 \\ -R & 0 & (R+3) & 3l \\ 0 & 2l^2 & 3l & 7l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$

$$\bar{u}_2 = \bar{u}_3 \quad \therefore \text{add col 3 to 1}$$

$$\begin{Bmatrix} \frac{Pl^3}{EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 12 & -6l & 0 \\ -6l & 8l^2 & 2l^2 \\ 3 & 0 & 3l \\ 3l & 2l^2 & 7l^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \quad \begin{array}{l} \frac{Pl^3}{EI} = 12\bar{u}_2 - 6l\theta_2 \quad (1) \\ -6l\bar{u}_2 + 8l^2\theta_2 + 2l^2\theta_3 = 0 \quad (2) \\ 3\bar{u}_2 + 3l\theta_3 = 0 \quad (3) \\ 3l\bar{u}_2 + 2l^2\theta_2 + 7l^2\theta_3 = 0 \quad (4) \end{array} \quad \begin{array}{l} \text{Eq. no.} \\ (1) \\ (2) \\ (3) \\ (4) \end{array}$$

$$2 \times (4) + (2) = 0u_2 + 12l^2\theta_2 + 16l^2\theta_3 = 0 \quad \therefore \theta_2 = -\frac{4}{3}\theta_3$$

$$(3) \text{ gives } \bar{u}_2 = -l\theta_3 \quad \therefore = \frac{3}{4}l\theta_2$$

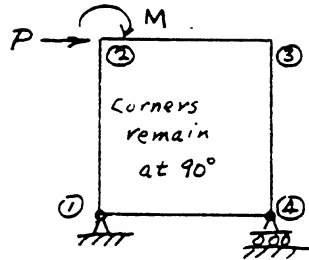
$$\text{Subst } \theta_2 = -\frac{4}{3}\theta_3 \text{ into (1)}$$

$$\frac{Pl^3}{EI} = 12u_2 - 6l\left(-\frac{4}{3}\theta_3\right) = 12\bar{u}_2 + 8l\theta_3 = 12\bar{u}_2 + 8\bar{u}_2 = 4\bar{u}_2$$

$$\therefore \bar{u}_2 = \frac{Pl^3}{4EI}, \quad \theta_3 = -\frac{Pl^2}{4EI}, \quad \theta_2 = \frac{Pl^2}{3EI}$$

Comparing with Prob(10-22(a)) the deflection under P is larger with pinned end, and the two rotations are dissimilar, and also larger. Conclusion: Pinned end leads to more flexible structure.

10-22 (c)



Element

1-2	$C = 0, S = 1$
2-3	$C = 1, S = 0$
3-4	$C = 0, S = -1$
4-1	$C = -1, S = 0$

Element stiffness

$$\textcircled{1}-\textcircled{2} \quad \begin{bmatrix} 12 & 0 & -6l & -12 & 0 & -6l \\ 0 & R & 0 & 0 & -R & 0 \\ -6l & 0 & 4l^2 & 6l & 0 & 2l^2 \\ -12 & 0 & 6l & 12 & 0 & 6l \\ 0 & -R & 0 & 0 & R & 0 \\ -6l & 0 & 2l^2 & 6l & 0 & 4l^2 \end{bmatrix}$$

$$\textcircled{2}-\textcircled{3} \quad \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} = \textcircled{1}-\textcircled{4}$$

$$\textcircled{3}-\textcircled{4} \quad \begin{bmatrix} 12 & 0 & 6l & -12 & 0 & 6l \\ 0 & R & 0 & 0 & -R & 0 \\ 6l & 0 & 4l^2 & -6l & 0 & 2l^2 \\ -12 & 0 & -6l & 12 & 0 & -6l \\ 0 & -R & 0 & 0 & R & 0 \\ 6l & 0 & 2l^2 & -6l & 0 & 4l^2 \end{bmatrix}$$

$$\textcircled{4}-\textcircled{1} \quad \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & -6l & 0 & -12 & -6l \\ 0 & -6l & 4l^2 & 0 & 6l & 2l^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & 6l & 0 & 12 & 6l \\ 0 & -6l & 2l^2 & 0 & 6l & 4l^2 \end{bmatrix}$$

Displacements.

$$\bar{u}_1 = \bar{v}_1 = 0$$

$$\bar{v}_2 = \bar{v}_3 = 0$$

$$\bar{u}_2 = \bar{u}_3$$

$$\bar{v}_4 = 0$$

$$\bar{u}_4 = 0$$

Coordinates not zero

$$\bar{u}_2, \bar{u}_3, \theta_1, \theta_2, \theta_3, \theta_4$$

none of the θ_i need to

be equal, however

we can assume that $\bar{u}_2 = \bar{u}_3$

This leaves 5 unknowns.

10-22 (c) Cont.

Assembled matrix (symmetric)

$$\begin{array}{c}
 \textcircled{1} \\
 \textcircled{2} \\
 \textcircled{3} \\
 \textcircled{4}
 \end{array}
 \left[\begin{array}{cccccc|cccc|cccc|cccc}
 (12+R) & 0 & -6l & -12 & 6 & -6l & -R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & (12+R) & 6l & 0 & -R & 0 & 0 & -12 & 6l & 0 & 0 & 0 & 0 & 0 & 0 \\
 -6l & 6l & 8l^2 & 6l & 0 & 2l^2 & 0 & -6l & 2l^2 & \theta_1 & - & - & - & - & - \\
 -12 & 0 & 6l & (12+R) & 0 & 6l & -R & 0 & 0 & u_2 & - & - & - & - & - \\
 0 & -R & 0 & 0 & (12+R) & 6l & 0 & -12 & 6l & 0 & 0 & 0 & 0 & 0 & 0 \\
 -6l & 0 & 2l^2 & 6l & 6l & 8l^2 & 0 & -6l & 2l^2 & \theta_2 & - & - & - & - & - \\
 & & & -R & 0 & 0 & (12+R) & 0 & 6l & -12 & 0 & 6l & \bar{u}_3 = \bar{u}_2 & - & - \\
 & & & 0 & -12 & -6l & 0 & (12+R) & -6l & 0 & -R & 0 & 0 & \bar{N}_3 = 0 & - \\
 & & & 0 & 6l & 2l^2 & 6l & -6l & 8l^2 & -6l & 0 & 2l^2 & \theta_3 & \theta_3 & - \\
 -R & 0 & 0 & & & & -12 & 0 & -6l & (12+R) & 0 & -6l & 0 & \bar{u}_4 = 0 & - \\
 0 & -12 & -6l & & & & 0 & -R & 0 & 0 & (12+R) & -6l & 0 & \bar{N}_4 = 0 & - \\
 0 & 6l & 2l^2 & & & & 6l & 0 & 2l^2 & -6l & -6l & 8l^2 & \theta_4 & \theta_4 & -
 \end{array} \right]$$

$\bar{u}_3 = \bar{u}_2$ ∴ add col. 7 to 4 and rewrite

$$\left\{ \begin{array}{c} \bar{u}_3 \\ F_{13} \\ F_{14} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} = \frac{EI}{l^3} \left[\begin{array}{ccccc} -6l & -12 & -6l & 0 & 0 \\ 6l & 0 & 0 & 0 & 6l \\ 8l^2 & 6l & 2l^2 & 0 & 2l^2 \\ 6l & 12 & 6l & 0 & 0 \\ 0 & 0 & 6l & 6l & 0 \\ 2l^2 & -6l & 8l^2 & 2l^2 & 0 \\ 0 & 12 & 0 & 6l & 6l \\ 0 & 0 & -6l & -6l & 0 \\ 0 & 6l & 2l^2 & 8l^2 & 2l^2 \\ 0 & -12 & 0 & -6l & -6l \\ -6l & 0 & 0 & 0 & -6l \\ 2l^2 & 6l & 0 & 2l^2 & 8l^2 \end{array} \right] \left\{ \begin{array}{c} \theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array} \right\}$$

12x5 matrix

5th and 8th equations are identical — both give $\theta_2 = -\theta_3$
 (remove 5th & 8th eqs. after subtracting
 3rd col from 3rd col.)

10-22 (c) Cont

Rewrite eq with col 4 subtracted from col 3 and 5th & 8th eqs. removed

$$\begin{Bmatrix} \bar{F}_{1x} \\ \bar{F}_{1y} \\ 0 \\ P \\ -M \\ 0 \\ 0 \\ \bar{F}_{4x} \\ \bar{F}_{4y} \\ 0 \end{Bmatrix} = \frac{\bar{EI}}{l^3} \begin{bmatrix} -6l & -12 & -6l & 0 \\ 6l & 0 & 0 & 6l \\ 8l^2 & 6l & 2l^2 & 2l^2 \\ 6l & 12 & 6l & 0 \\ 2l^2 & 6l & 6l^2 & 0 \\ 0 & 12 & -6l & 6l \\ 0 & 6l & -6l^2 & 2l^2 \\ 0 & -12 & 6l & -6l \\ -6l & 0 & 0 & -6l \\ 2l^2 & 6l & -2l^2 & 8l^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \theta_4 \end{Bmatrix} \quad 16 \times 4 \text{ matrix}$$

Eqs. for \bar{F}_{1y} and \bar{F}_{4y} are equal but of opposite signs

$$\therefore \bar{F}_{1y} = -\bar{F}_{4y} \quad (\text{remove these two eqs.})$$

6th eq and \bar{F}_{4x} are of opposite signs but 6th eq = 0

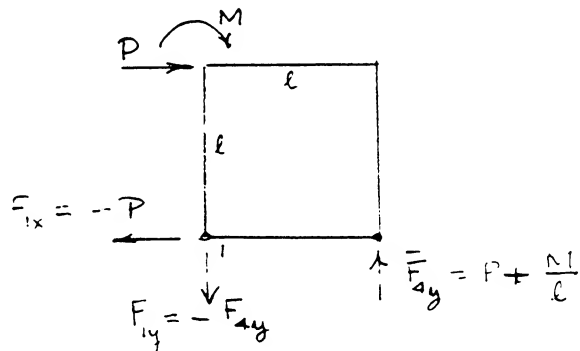
$$\therefore \bar{F}_{4x} = 0 \quad (\text{remove these two eqs.})$$

Eqs for \bar{F}_{1x} and P are of opposite signs

$$\therefore F_{1x} = -P$$

Moment about ① gives

$$\bar{F}_{4y} l = Pl + M$$



The above is just for checking, since these results can be obtained from statics.

10-22 (c) Cont.

Rewrite unused eqs from 10×4 matrix

$$\frac{Pl^3}{EI} = 6l\theta_1 + 12\bar{u}_2 + 6l\theta_2 \quad (1)$$

$$-\frac{Ml^2}{EI} = 2l\theta_1 + 6\bar{u}_2 + 6l\theta_2 \quad (2)$$

$$0 = 8l^2\theta_1 + 6l\bar{u}_2 + 2l^2\theta_2 + 2l^2\theta_4 \quad (3)$$

$$0 = 2l^2\theta_1 + 6l\bar{u}_2 - 2l^2\theta_2 + 8l^2\theta_4 \quad (4)$$

Eliminate θ_4 betw. (3) & (4)

$$0 = 30l^2\theta_1 + 18l\bar{u}_2 + 10l^2\theta_2 \quad (5)$$

Eqs. (1) (2) & (5) has 3 unknowns with 3 eqs.

In (5) solve for $l\theta_1$ in terms of \bar{u}_2 and θ_2

$$l\theta_1 = -(.60\bar{u}_2 + .3333l\theta_2) \text{ subst. into (1) and (2)}$$

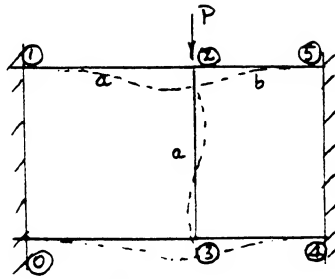
$$\frac{Pl^3}{EI} = (12 - 3.60)\bar{u}_2 + (6l - 2l)\theta_2 = 8.40\bar{u}_2 + 4l\theta_2$$

$$-\frac{Ml^2}{EI} = (6 - 1.20)\bar{u}_2 + (6l - .666l)\theta_2 = 4.80\bar{u}_2 + 5.333l\theta_2$$

Eliminate θ_2 for eq. for \bar{u}_2 , then eliminate \bar{u}_2

$$\boxed{\begin{aligned}\bar{u}_2 &= \frac{1}{4.80} \left(\frac{Pl^3}{EI} + .75 \frac{Ml^2}{EI} \right) \\ \theta_2 &= \frac{-1}{5.333l} \left(\frac{Pl^3}{EI} + 1.75 \frac{Ml^2}{EI} \right)\end{aligned}}$$

10-22 (d)



$$\text{let } a = l_1 = l$$

$$b = l_2 = \frac{b}{a} l_1$$

$$\bar{u}_2 = \bar{u}_3 = 0$$

$$\bar{v}_2 = \bar{v}_3$$

$$\left\{ \begin{array}{l} \bar{u}_1 = 0 \\ \bar{v}_1 = 0 \\ \theta_1 = 0 \\ \vdots \\ \bar{u}_2 = 0 \\ \bar{v}_2 = 0 \\ \theta_2 \\ \hline \bar{u}_3 = 0 \\ \bar{v}_3 = 0 \\ \theta_3 \\ \hline \bar{u}_4 = 0 \\ \bar{v}_4 = 0 \\ \theta_4 = 0 \end{array} \right.$$

$$\bar{k}_{1-2} = \frac{EI}{l_1^3} \begin{bmatrix} -R & 0 & 0 \\ 0 & -12 & 6l \\ 0 & 6l & 2l^2 \\ R & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix} = \bar{k}_{0-3}$$

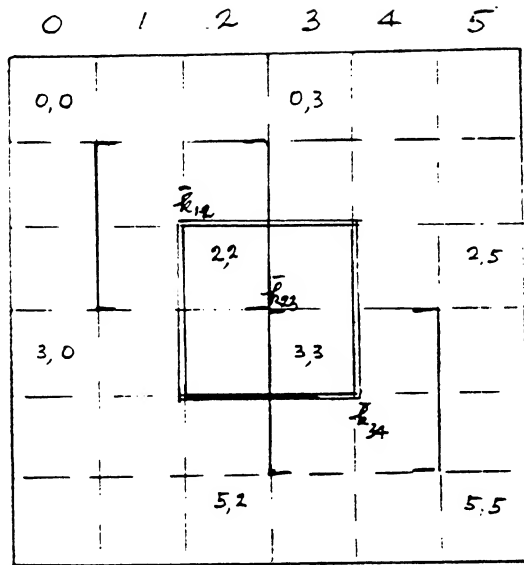
$$\bar{k}_{2-3} = \frac{EI}{l_1^3} \begin{bmatrix} 12 & 0 & 6l & -12 & 0 & 6l \\ 0 & R & 0 & 0 & -R & 0 \\ 6l & 0 & 4l^2 & -6l & 0 & 2l^2 \\ -12 & 0 & -6l & 12 & 0 & -6l \\ 0 & -R & 0 & 0 & R & 0 \\ 6l & 0 & 2l^2 & -6l & 0 & 4l^2 \end{bmatrix}$$

$$\bar{k}_{3-4} = \frac{EI}{l_2^3} \begin{bmatrix} R & 0 & 0 \\ 0 & 12 & 6l_2 \\ 0 & 6l_2 & 4l_2^2 \\ -R & 0 & 0 \\ 0 & -12 & -6l_2 \\ 0 & 6l_2 & 2l_2^2 \end{bmatrix} = \frac{EI}{l^3} \begin{bmatrix} R(\frac{a}{b})^3 & 0 & 0 \\ 0 & 12(\frac{a}{b})^3 & 6l(\frac{a}{b})^2 \\ 0 & 6l(\frac{a}{b})^2 & 4l^2(\frac{a}{b}) \\ -R(\frac{a}{b})^3 & 0 & 0 \\ 0 & -12(\frac{a}{b})^2 & -6l(\frac{a}{b})^2 \\ 0 & 6l(\frac{a}{b})^2 & 2l^2(\frac{a}{b}) \end{bmatrix} = \bar{k}_{2-5}$$

$$= \bar{k}_{3-4}$$

For the assembly we can first consider a matrix space of 6×6 each of which is composed of a 3×3 elements. i.e. \bar{k}_{2-3} has 4 sub-matrices, each corner of which has a 3×3 matrix. Since ①, ②, ④ and ⑤ all have zero deflections, we will be concerned only with the displacements of ② and ③ which occupies the central portion of the 6×6 space.

10-22 (d) Cont.



sub matrices of \bar{k}_{2-5}

2,2 2,5

5,2 5,5

shown in 6x6 space

sub matrices of \bar{k}_{0-3}

0,0 0,3

3,0 3,3

shown in 6x6 space

The matrix of interest which is indicated by the square in double lines also involves the lower right quarter of \bar{k}_{12} , all of \bar{k}_{23} and upper left quarter of \bar{k}_{3-4}

$$EI \begin{bmatrix} R+12 & 0+0 & 0+6l & -12 & 0 & 6l \\ +R(\frac{a}{b})^3 & 0 & 0 & 0 & 0 & 0 \\ 0+0 & 12+R & -6l+0 & 0 & -R & 0 \\ 0 & 12(\frac{a}{b})^3 & 6l(\frac{a}{b})^2 & 0 & 0 & 0 \\ 0+6l & -6l+0 & 4l^2+4l^2 & -6l & 0 & 2l^2 \\ 0 & 6l(\frac{a}{b})^2 & 4l^2(\frac{a}{b}) & 0 & 0 & 0 \\ -12 & 0 & -6l & R+12 & 0+0 & 0-6l \\ 0 & -R & 0 & 0+0 & 12+R & -6l+0 \\ 6l & 0 & 2l^2 & 0-6l & -6l+0 & 4l^2+4l^2 \\ & & & R(\frac{a}{b})^3 & 12(\frac{a}{b})^3 & 6l(\frac{a}{b})^2 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix}$$

10-22 (d) Cont

Note that
$$\begin{cases} \bar{u}_2 = \bar{u}_3 = 0 \\ \bar{v}_2 = \bar{v}_3 \\ \theta_2 = \theta_3 \end{cases}$$

cross out cols. 1 and 4

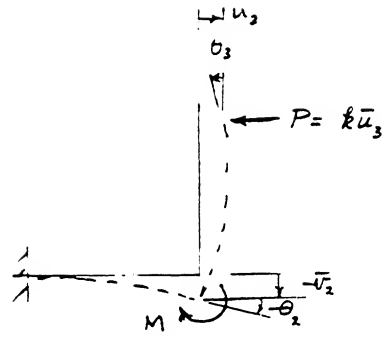
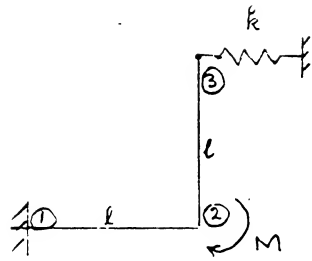
add col. 5 to 2

add col. 6 to 3

Resulting equations

$$\begin{cases} \bar{F}_{2y} = -P \\ \bar{M}_2 = 0 \end{cases} = \frac{EI}{\alpha^3} \begin{bmatrix} (12)\left[1 + \left(\frac{a}{b}\right)^3\right] & 6\ell\left[\left(\frac{a}{b}\right)^2 - 1\right] \\ 6\ell\left[\left(\frac{a}{b}\right)^2 - 1\right] & 4\ell^2\left[2.5 + \left(\frac{a}{b}\right)\right] \end{bmatrix} \begin{cases} \bar{v}_2 \\ \theta_2 \end{cases}$$

10-22 (c)



$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F_{3x} \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & R+12 & 0+0 & 0-6l \\ 0 & -12 & -6l & 0+0 & 12+R & -6l+0 \\ 0 & 6l & 2l^2 & 0-6l & -6l+0 & 4l^2+4l^2 \\ -12 & 0 & 6l & (12+\frac{k l^3}{EI}) & 0 & 6l \\ 0 & -R & 0 & 0 & R & 0 \\ -6l & 0 & 2l^2 & 6l & 0 & 4l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1=0 \\ \bar{v}_1=0 \\ \bar{\theta}_1=0 \\ \bar{u}_2=0 \\ \bar{v}_2 \\ \bar{\theta}_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \bar{\theta}_3 \end{Bmatrix}$$

$\bar{v}_3 = \bar{v}_2 \therefore$ add col. 8 to col. 5

$$\begin{Bmatrix} 0 \\ -M \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l & 0 & 0 \\ -6l & 8l^2 & 6l & 2l^2 \\ 0 & 6l & (12+\frac{k l^3}{EI}) & 6l \\ 0 & 2l^2 & 6l & 4l^2 \end{bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \bar{\theta}_2 \\ \bar{u}_3 \\ \bar{\theta}_3 \end{Bmatrix} \quad \text{Sym. } 4 \times 4$$

4 eqs, 4 unknowns

10-22 (e) Cont.

$$\textcircled{1} \quad 0 = 12 \bar{v}_2 - 6l\theta_2$$

$$\textcircled{2} \quad -\frac{Ml^3}{EI} = -6l\bar{v}_2 + 8l^2\theta_2 + 6l\bar{u}_3 + 2l^2\theta_3$$

$$\textcircled{3} \quad 0 = 6l\theta_2 + \left(12 + \frac{kl^3}{EI}\right)\bar{u}_3 + 6l\theta_3$$

$$\textcircled{4} \quad 0 = 2l^2\theta_2 + 6l\bar{u}_3 + 4l^2\theta_3$$

From $\textcircled{1}$ $\bar{v}_2 = \frac{l}{2}\theta_2$ sub into $\textcircled{2}$

$$\textcircled{2'} \quad -\frac{Ml^3}{EI} = 5l^2\theta_2 + 6l\bar{u}_3 + 2l^2\theta_3$$

From $\textcircled{4}$ $\theta_2 = -\frac{3}{l}\bar{u}_3 - 2\theta_3$

Sub. into $\textcircled{3}$ $0 = -6l\left(\frac{3}{2}\bar{u}_3 + 2\theta_3\right) + \left(12 + \frac{kl^3}{EI}\right)\bar{u}_3 + 6l\theta_3$
 $= (-18 + 12 + \frac{kl^3}{EI})\bar{u}_3 - 12l\theta_3 + 6l\theta_3$

$$\therefore \underline{6l\theta_3 = \left(-6 + \frac{kl^3}{EI}\right)\bar{u}_3}$$

$$-\frac{Ml^3}{EI} = 5l^2\left(-\frac{3}{l}\bar{u}_3 - 2\theta_3\right) + 6l\bar{u}_3 + 2l^2\theta_3 = -9l\bar{u}_3 - 8l^2\theta_3$$

Sub for θ_3 in above eq.

$$-\frac{Ml^3}{EI} = -9l\bar{u}_3 - 8l^2\left(\frac{-6 + \frac{kl^3}{EI}}{6l}\right)\bar{u}_3 = \left(-l - \frac{4}{3}\frac{kl^3}{EI}\right)\bar{u}_3$$

$$\therefore \boxed{\bar{u}_3 = \frac{Ml^2}{\left(1 + \frac{4}{3}\frac{kl^3}{EI}\right)EI}}$$

$$\boxed{\theta_3 = \frac{1}{6l} \frac{\left(-6 + \frac{kl^3}{EI}\right)}{\left(1 + \frac{4}{3}\frac{kl^3}{EI}\right)} \frac{Ml^2}{EI}}$$

$$\boxed{\theta_2 = -\frac{3}{l}\bar{u}_3 - 2\theta_3}$$

$$\boxed{\bar{v}_2 = -\frac{3}{2}\bar{u}_3 - l\theta_2}$$

10-22 (f) cont.

Eliminate terms with R by subtracting 2nd row from row 1.

From 4th row $\theta_3 = -\frac{2}{l} \bar{u}_2$

From 1st row after eliminating R and θ_3

$$\frac{Pl^3}{EI} = 16.20 \bar{u}_2 - 4.244 \bar{v}_2 - 3l \theta_2$$

5th row $0 = 12 \bar{u}_2 - 12 \bar{v}_2 - 6l \theta_2$ x by $\frac{16.2}{12}$ & subtr.

$$\frac{Pl^3}{EI} = 0 + 11.95 \bar{v}_2 + 5.1l \theta_2 \quad (a)$$

3rd row $1.5l \bar{u}_2 + 4.5l \bar{v}_2 + 5.41l^2 \theta_2 + \underset{\substack{\uparrow \\ \text{elim}}}{2l^2 \theta_3} = 0$

betw 5th + 3rd rows

$$6 \bar{v}_2 + 15.66l \theta_2 = 0$$

$$\therefore \bar{v}_2 = -2.61l \theta_2 \quad (b)$$

(b) into (a) and add

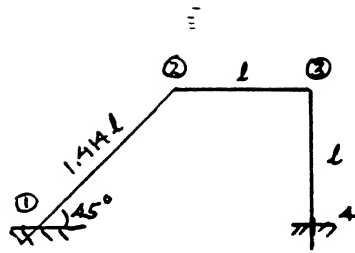
$$\theta_2 = \frac{-1}{26.097} \frac{Pl^2}{EI} = -0.0383 \frac{Pl^2}{EI}$$

$$\bar{v}_2 = .10 \frac{Pl^3}{EI}$$

$$\bar{u}_2 = .0808 \frac{Pl^3}{EI}$$

$$\theta_3 = -.1616 \frac{Pl^2}{EI}$$

10-2) (f)



Element 1-2

$$\alpha = 45^\circ$$

$$C = .707$$

$$S = .707$$

$$\left(\frac{EI}{(1.414l)^3} \right) \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} = \left(\frac{EI}{l^3} \right) \begin{bmatrix} (.1769R+2.122)(.1769R-2.122) & 1.5l \\ (.1769R-2.122)(.1769R+2.122) & -1.5l \\ 1.5l & -1.5l & 1.414l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$$

Element 2-3

$$\alpha = 0$$

$$C = 1$$

$$S = 0$$

$$\left(\frac{EI}{l^3} \right) \begin{bmatrix} R & 0 & 0 & -R & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & R & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Element 3-4

$$\alpha = -90^\circ$$

$$C = 0$$

$$S = -1$$

$$\left(\frac{EI}{l^3} \right) \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ \theta_3 \end{Bmatrix} = \left(\frac{EI}{l^3} \right) \begin{bmatrix} 12 & 0 & 6l \\ 0 & R & 0 \\ 6l & 0 & 4l^2 \end{bmatrix}$$

Assembled stiffness matrix & equation

$$\begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \left(\frac{EI}{l^3} \right) \begin{bmatrix} (1.1769R+2.122)(.1769R-2.122) & 1.5l & -R & 0 & 0 \\ (.1769R-2.122)(.1769R+2.122) & 4.5l & 0 & -12 & 6l \\ 1.5l & 4.5l & 5.414l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & (R+12) & 0 & 6l \\ 0 & -12 & -6l & 0 & (R+12) & -6l \\ 0 & 6l & 2l^2 & 6l & -6l & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

$\bar{u}_3 = \bar{u}_2$
 $\bar{v}_3 = 0$

$$\frac{Pl}{EI} = (.1769R+2.122)\bar{u}_2 + (.1769R-2.122)\bar{v}_2 + 1.5l\theta_2$$

$$0 = (.1769R-2.122)\bar{u}_2 + (.1769R+2.122)\bar{v}_2 + 4.5l\theta_2 + 6l\theta_3$$

$$0 = 1.5l\bar{u}_2 + 4.5l\bar{v}_2 + 5.414l^2\theta_2 + 2l^2\theta_3$$

$$0 = 12\bar{u}_2 + 6l\theta_3$$

$$0 = -12\bar{v}_2 - 6l\theta_2 - 6l\theta_3$$

$$0 = 6l\bar{u}_2 + 6l\bar{v}_2 + 2l^2\theta_2 + 8l^2\theta_3$$

10-23 Refer to prob 10-17 with stiffness equation

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6 & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

The moment M_1 at the pin is now altered to $M_1 - K\theta_1$, where K is the torsional spring stiffness at the pin. By transferring the term $-K\theta_1$ into the matrix, the only term altered in the equation is the element (2,2) which becomes $(4l^2 + K\frac{l^3}{EI})$.

To evaluate K , the data specifies the following

$$(mgl)\frac{l}{2} = K\theta_1 = K\frac{1}{10} \quad \therefore K = 5mgl^2$$

It is also suggested to choose

$$\frac{5mgl^3}{EI} = K\frac{l}{EI} = 1.0 \quad \text{or} \quad \frac{Kl^3}{EI} = 1 \text{ } l^2$$

Thus the (2,2) term becomes $(4l^2 + 1l^2) = 5l^2$

Whether the problem is solved as a single element or a double element problem, only this one term is changed for the 1st element, which will increase the natural frequencies of the system. Following comparison is given for the single element beam

$$K = 0 \quad \omega_1 = 0 \quad \omega_2 = 17.54 \sqrt{\frac{EI}{ml^2}}$$

$$K\frac{l^3}{EI} = 1.0 \quad \omega_1 = 1.56 \sqrt{\frac{EI}{ml^2}} \quad \omega_2 = 19.02 \sqrt{\frac{EI}{ml^2}}$$

10 -23 cont

```
»m=[4 13 -3;13 156 -22; -3 -22 4]
```

m =

4	13	-3
13	156	-22
-3	-22	4

```
»k=[5 -6 2; -6 12 -6; 2 -6 4]
```

k =

5	-6	2
-6	12	-6
2	-6	4

```
»a=inv(m)*k
```

a =

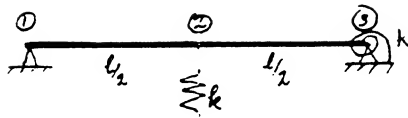
5.0000	-9.0000	4.7143
0.6429	-1.5000	0.9286
7.7857	-16.5000	9.6429

```
»eig(a)
```

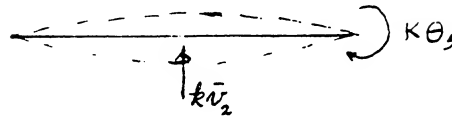
ans =

12.2754
0.8616
0.0058

10-24 and -25



horiz. displ. = 0
 $\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$



Element Stiffness 1-2 and 2-3 Eq. (10.2-1) with $l = \frac{l}{2}$

$$\frac{8EI}{l^3} \begin{bmatrix} 12 & 3l & -12 & 3l \\ 3l & l^2 & -3l & \frac{1}{2}l^2 \\ -12 & -3l & 12 + \frac{k l^3}{8EI} & -3l \\ 3l & \frac{1}{2}l^2 & -3l & l^2 + \frac{k l^3}{8EI} \end{bmatrix} \text{ displ.} = \begin{Bmatrix} \bar{v}_1 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \text{ or } \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$$

The term $\frac{k l^3}{8EI}$ is used only in one of the \bar{v}_2 elements

Element Mass 1-2 and 2-3 Eq. (10.2-10) with $l = \frac{l}{2}$

$$\frac{m l}{840} \begin{bmatrix} 156 & 11l & 54 & -6.5l \\ 11l & l^2 & 6.5l & -7.5l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -7.5l^2 & -11l & l^2 \end{bmatrix}$$

Assembled

$$\frac{8EI}{l^3} \begin{bmatrix} 12 & 3l & -12 & 3l & & \\ 3l & l^2 & -3l & \frac{1}{2}l^2 & & \\ -12 & -3l & 12 + \frac{k l^3}{8EI} & -3l & & \\ 3l & \frac{1}{2}l^2 & -3l & l^2 + \frac{k l^3}{8EI} & & \\ & & & & 12 & 3l \\ & & & & 3l & \frac{1}{2}l^2 \end{bmatrix} \text{ displ.} = \begin{Bmatrix} \bar{v}_1 = 0 \\ \theta_1 \\ \bar{v}_2 \\ \theta_2 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

∴ delete cols 1 and 5 and rows 1 and 5

10-21-25 Cont.

Assembled:

$$\frac{ml}{840} \begin{bmatrix} 156 & 11l & 54 & -6.5l \\ 11l & l^2 & 6.5l & -7.5l^2 \\ 54 & 6.5l & 312 & 0 \\ -6.5l & -7.5l^2 & 0 & 2l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -7.5l^2 & 11l & l^2 \end{bmatrix}$$

Eq. of motion is a 4×4

$$\left[-\frac{\omega^2 ml}{840} \begin{bmatrix} l^2 & 6.5l & -7.5l^2 & 0 \\ 6.5l & 312 & 0 & -6.5l \\ -7.5l^2 & 0 & 2l^2 & -7.5l^2 \\ 0 & -6.5l & -7.5l^2 & l^2 \end{bmatrix} + \frac{8EI}{l^3} \begin{bmatrix} l^2 & -3l & .5l^2 & 0 \\ -3l & (24 + \frac{kl^3}{8EI}) & 0 & 3l \\ .5l^2 & 0 & 2l^2 & .5l^2 \\ 0 & 3l & .5l^2 & (l^2 + \frac{kl^3}{8EI}) \end{bmatrix} \right] \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left. \begin{aligned} \frac{kl^3}{EI} &= \frac{1}{2} \\ \frac{kl^3}{EI} &= \frac{1}{4} l^2 \end{aligned} \right\} \text{for computer solution let } l=1 \text{ inside the matrices}$$

$$\lambda = \frac{\omega^2 ml^4}{6720 EI} \quad \omega = \sqrt{6720 \lambda} \sqrt{\frac{EI}{ml^4}}$$

10 - 25 cm
»m

with springs

m =

1.0000	6.5000	-0.7500	0
6.5000	312.0000	0	-6.5000
-0.7500	0	2.0000	-0.7500
0	-6.5000	-0.7500	1.0000

»k

k =

1.0000	-3.0000	0.5000	0
-3.0000	24.0600	0	3.0000
0.5000	0	2.0000	0.5000
0	3.0000	0.5000	1.0310

»a

a =

1.0000	-0.5784	1.2239	0.5916
-0.5784	0.3904	-0.6436	-0.0179
1.2239	-0.6436	2.7582	2.2054
0.5916	-0.0179	2.2054	4.0139

»[v,d]=eig(a)

v =

0.2478	0.5229	0.6611	-0.4776
-0.0977	-0.4054	-0.2697	-0.8680
0.6131	0.4508	-0.6439	-0.0795
0.7438	-0.5991	0.2750	0.1106

$$\omega = \sqrt{6720} \sqrt{\frac{EI}{mL^4}}$$

d =

6.0312	0	0	0
0	1.8257	0	0
0	0	0.2901	0
0	0	0	0.0155

$$\omega = \begin{matrix} 10.19 \\ 44.15 \\ 110.8 \\ 201.3 \end{matrix}$$

10-25 cont
 $\gg m$

$m =$

1.0000	6.5000	-0.7500	0
6.5000	312.0000	0	-6.5000
-0.7500	0	2.0000	-0.7500
0	-6.5000	-0.7500	1.0000

$k =$

1.0000	-3.0000	0.5000	0
-3.0000	24.0000	0	3.0000
0.5000	0	2.0000	0.5000
0	3.0000	0.5000	1.0000

$\gg a = q_{in}' * k * q_{in}$

$a =$

1.0000	-0.5784	1.2239	0.5916
-0.5784	0.3902	-0.6435	-0.0180
1.2239	-0.6435	2.7582	2.2055
0.5916	-0.0180	2.2055	3.9571

$\gg [v, d] = \text{eig}(a)$

$v =$

0.2500	0.5246	0.6614	-0.4742
-0.0989	-0.4082	-0.2618	-0.8689
0.6171	0.4430	-0.6449	-0.0841
0.7395	-0.6016	0.2795	0.1142

$d =$

6.0000	0	0	0
0	1.8052	0	0
0	0	0.2857	0
0	0	0	0.0146

Without Springs.

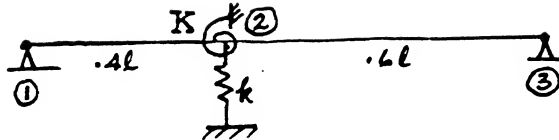
$$\omega = \sqrt{67201} \sqrt{\frac{EI}{\lambda^4}}$$

exact
 $\omega = 9.908$
 43.817
 110.13
 200.8

9.87
 39.48
 88.5
 156

10-25

Stiffness Matrix



Element ① to ② $l_1 = .4l$, $u_1 = 0$

$$\frac{EI}{l_1^3} \begin{bmatrix} 4l_1^2 & -6l_1 & 2l_1^2 \\ -6l_1 & 12 & -6l_1 \\ 2l_1^2 & -6l_1 & 4l_1^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \frac{EI}{.064l^3} \begin{bmatrix} 4(.16l^2) & -6(.4l) & 2(.16l^2) \\ -6(.4l) & 12 & -6(.4l) \\ 2(.16l^2) & -6(.4l) & 4(.16l^2) \end{bmatrix}$$

(let $l = 1$.)

$$= \frac{EI}{l^3} \begin{bmatrix} 10 & -37.5 & 5 \\ -37.5 & 187.5 & -37.5 \\ 5 & -37.5 & 10 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

Element ② to ③ $l_2 = .6l$, $u_3 = 0$

$$\frac{EI}{l_2^3} \begin{bmatrix} 12 & 6l_2 & 6l_2 \\ 6l_2 & 4l_2^2 & 2l_2^2 \\ 6l_2 & 2l_2^2 & 4l_2^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{EI}{.216l^3} \begin{bmatrix} 12 & 6(.6l) & 6(.6l) \\ 6(.6l) & 4(.36l^2) & 2(.36l^2) \\ 6(.6l) & 2(.36l^2) & 4(.36l^2) \end{bmatrix}$$

$$= \frac{EI}{l^3} \begin{bmatrix} 55.56 & 16.67 & 16.67 \\ 16.67 & 6.67 & 3.33 \\ 16.67 & 3.33 & 6.67 \end{bmatrix} \begin{Bmatrix} u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Assemble

$$\frac{EI}{l^3} \begin{bmatrix} 10 & -37.5 & 5 & 0 \\ -37.5 & (187.5) & (-37.5) & 16.67 \\ 5 & (-37.5) & (10) & 3.33 \\ 0 & 16.67 & 3.33 & 6.67 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$= \frac{EI}{l^3} \begin{bmatrix} 10 & -37.5 & 5 & 0 \\ -37.5 & 243.1 & -20.83 & 16.67 \\ 5 & -20.83 & 16.67 & 3.33 \\ 0 & 16.67 & 3.33 & 6.67 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

must add $\frac{kl^3}{EI} = 1$ to (2,2) and $\frac{Kl^3}{EI} = 2$ to (3,3)

10-25 Cont.

Mass Matrix

Element ① to ② $l_1 = .4l$, $U_1 = 0$

$$\frac{ml}{420} \begin{bmatrix} 4l_1^2 & 13l_1 & -3l_1^2 \\ 13l_1 & 156 & -22l_1 \\ -3l_1^2 & -22l_1 & 4l_1^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ U_2 \\ \theta_2 \end{Bmatrix} = \frac{m \cdot 4l}{420} \begin{bmatrix} 4(.16) & 13(.4) & -3(.16) \\ 13(.4) & 156 & -22(.4) \\ -3(.16) & -22(.4) & 4(.16) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ U_2 \\ \theta_2 \end{Bmatrix}$$

$$= \frac{ml}{420} \begin{bmatrix} .256 & 2.08 & -.192 \\ 2.08 & 62.4 & -3.52 \\ -.192 & -3.52 & .256 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ U_2 \\ \theta_2 \end{Bmatrix}$$

Element ② to ③ $l_2 = .6l$, $U_3 = 0$

$$\frac{.6ml}{420} \begin{bmatrix} 156 & 22l_2 & -13l_2 \\ 22l_2 & 4l_2^2 & -3l_2^2 \\ -13l_2 & -3l_2^2 & 4l_2^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{ml}{420} \begin{bmatrix} 93.6 & 7.92 & -4.68 \\ 7.92 & .864 & -.648 \\ -4.68 & -.648 & .864 \end{bmatrix} \begin{Bmatrix} U_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Assemble

$$\frac{ml}{420} \begin{bmatrix} .256 & 2.08 & -.192 & 0 \\ 2.08 & (62.4) & (-3.52) & -4.68 \\ -.192 & (-3.52) & (.256) & -.648 \\ 0 & -4.68 & -.648 & .864 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ U_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$= \frac{ml}{420} \begin{bmatrix} .256 & 2.08 & -.192 & 0 \\ 2.08 & 156 & 4.40 & -4.68 \\ -.192 & 4.40 & 1.20 & -.648 \\ 0 & -4.68 & -.648 & .864 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ U_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

Eq. of motion

$$-\lambda \begin{bmatrix} .256 & 2.08 & -.192 & 0 \\ 2.08 & 156 & 4.40 & -4.68 \\ -.192 & 4.40 & 1.20 & -.648 \\ 0 & -4.68 & -.648 & .864 \end{bmatrix} + \begin{bmatrix} 10 & -37.5 & 5 & 0 \\ -37.5 & 243.1 & -20.83 & 16.67 \\ 5 & -20.83 & 16.67 & 3.33 \\ 0 & 16.67 & 3.33 & 6.67 \end{bmatrix}$$

10-26 cont

```
>>m=[.2560 2.08 -.192 0; 2.08 156 4.4 -4.68;-.192 4.4 1.12 -.648;  
0 -4.68 -6.48 .864]
```

m =

0.2560	2.0800	-0.1920	0
2.0800	156.0000	4.4000	-4.6800
-0.1920	4.4000	1.1200	-0.6480
0	-4.6800	-6.4800	0.8640

```
>>k=[10 -37.5 5 0; -37.5 243.1 -20.83 16.67; 5 -20.83 16.67 3.333;  
0 16.67 3.333 6.67]
```

k =

10.0000	-37.5000	5.0000	0
-37.5000	243.1000	-20.8300	16.6700
5.0000	-20.8300	16.6700	3.3330
0	16.6700	3.3330	6.6700

without springs

```
>>q=chol(m)
```

q =

0.5060	4.1110	-0.3795	0
0	11.7941	0.5053	-0.3968
0	0	0.8489	-0.5271
0	0	0	0.6547

```
>>qin=inv(q)
```

qin =

1.9764	-0.6889	1.2936	0.6239
0	0.0848	-0.0505	0.0108
0	0	1.1780	0.9484
0	0	0	1.5273

```
>>a=qin'*k*qin
```

a =

39.0625	-19.8999	40.9488	20.9071
-19.8999	10.8744	-21.5071	-8.5659

10-26 cont

```
40.9488 -21.5071 63.0979 42.4767
20.9071 -8.5659 42.4767 49.6672
```

```
>>[v,d]=eig(a)
```

v =

```
-0.4726 -0.5505 -0.5870 -0.3591
0.2374 0.3491 0.0356 -0.9058
-0.6816 -0.0576 0.7086 -0.1730
-0.5056 0.7561 -0.3899 0.1436
```

d =

```
130.4892 0 0 0
0 27.2528 0 0
0 0 4.7246 0
0 0 0 0.2353
```

$$\lambda = \frac{\omega^2 m l^4}{420 EI}$$

$$\omega = \sqrt{420 \lambda} \sqrt{\frac{EI}{m l^4}}$$

$$\omega = \begin{matrix} 234 \\ 106.9 \\ 44.55 \\ 9.943 \end{matrix}$$

k =

```
10.0000 -37.5000 5.0000 0
-37.5000 244.1000 -20.8300 16.6700
5.0000 -20.8300 18.6700 3.3330
0 16.6700 3.3330 6.6700
```

with springs

```
>>a=qin'*k*qin
```

a =

```
39.0625 -19.8999 40.9488 20.9071
-19.8999 10.8816 -21.5114 -8.5650
40.9488 -21.5114 65.8757 44.7105
20.9071 -8.5650 44.7105 51.4662
```

10-26 cont

≡

```
>>[v,d]=eig(a)
```

v =

-0.4588	0.5716	-0.5727	0.3670
0.2303	-0.3566	0.0393	0.9046
-0.6875	0.0499	0.7057	0.1641
-0.5137	-0.7373	-0.4153	-0.1418

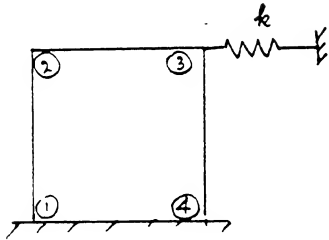
d =

133.8189	0	0	0
0	28.0864	0	0
0	0	5.1330	0
0	0	0	0.2477

$\omega =$ 237
108.6
44.44
10.20

10-27

Stiffness matrix



$$\frac{EI}{l^3} \begin{bmatrix} (12+R) & 0 & 6l & -R & 0 & 0 \\ 0 & (12+R) & 6l & 0 & -12 & 6l \\ 6l & 6l & 8l^2 & 0 & -6l & 2l^2 \\ -R & 0 & 0 & (12+R+\frac{kl}{3}) & 0 & 6l \\ 0 & -12 & -6l & 0 & (12+R) & -6l \\ 0 & 6l & 2l^2 & 6l & -6l & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 = 0 \\ \theta_2 \\ \bar{u}_3 = \bar{u}_2 \\ \bar{v}_3 = 0 \\ \theta_3 \end{Bmatrix}$$

Upper left qtr = lower right qtr of elem. ①-② plus upper left qtr of elem. ②-③.

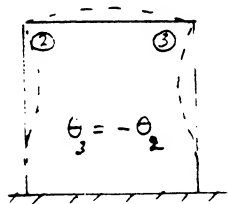
Lower right qtr = upper left qtr of elem. ③-④ plus lower right qtr of elem. ②-③.

mass matrix $\frac{ml}{420}$

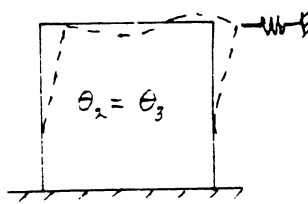
$$\begin{bmatrix} (156+N) & 0 & 22l & \frac{1}{2}N & 0 & 0 \\ 0 & (156+N) & 22l & 0 & -13l & 0 \\ 22l & 22l & 8l^2 & 0 & 13l & -3l^2 \\ \frac{1}{2}N & 0 & 0 & (156+N) & 0 & 22l \\ 0 & -13l & -3l^2 & 0 & (156+N) & -22l \\ 0 & -13l & -3l^2 & 22l & -22l & 8l^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{v}_3 \\ \ddot{\theta}_3 \end{Bmatrix}$$

Assumptions

$$\bar{u}_2 = \bar{u}_3, \quad \bar{v}_2 = \bar{v}_3 = 0, \quad \text{two modes} \begin{cases} \theta_3 = -\theta_2 \rightarrow \text{spring not involved} \\ \theta_3 = \theta_2 \end{cases}$$



Spring not in action



10-27 Cont.

General eq. of motion

$$\frac{ml}{420} \begin{bmatrix} (156+N) & 22l & \frac{1}{2}N & 0 \\ 22l & 8l^2 & 0 & -3l^2 \\ \frac{1}{2}N & 0 & (156+N) & 22l \\ 0 & -3l^2 & 22l & 8l^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{EI}{l^3} \begin{bmatrix} (12+R) & 6l & -R & 0 \\ 6l & 8l^2 & 0 & 2l^2 \\ -R & 0 & (12+R+\frac{kl^3}{EI}) & 6l \\ 0 & 2l^2 & 6l & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \bar{\theta}_2 \\ \bar{u}_3 \\ \bar{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Symmetric mode $\bar{u}_2 = \bar{u}_3 = 0 \quad \therefore \text{cross out col. ①, col. ③}$
 $\theta_3 = -\theta_2 \quad \text{row ①, row ③}$

$$\frac{ml}{420} \begin{bmatrix} 8l^2 & -3l^2 \\ -3l & 8l^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 8l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix} \begin{Bmatrix} \bar{\theta}_2 \\ \bar{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[-\omega^2 \frac{ml}{420} (11) + \frac{EI}{l^3} (6) \right] l^2 \bar{\theta}_2 = 0$$

$$\omega^2 = \frac{6 \times 420}{11} \frac{EI}{ml^4} \quad \omega = 15.14 \sqrt{\frac{EI}{ml^4}}$$

Antisym. mode $\bar{u}_2 = \bar{u}_3 \neq 0, \quad \theta_2 = \theta_3$

$$\left[-\frac{\omega^2 ml}{420} \begin{bmatrix} (156+\frac{3}{2}N) & 22l \\ 22l & 5l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} (12+\frac{kl^3}{EI}) & 6l \\ 6l & 10l^2 \end{bmatrix} \right] \begin{Bmatrix} \bar{u} \\ \bar{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

with $\lambda = \left(\frac{\omega^2 ml^4}{420 EI} \right) \quad N=140 \quad \text{Characteristic eq.} =$

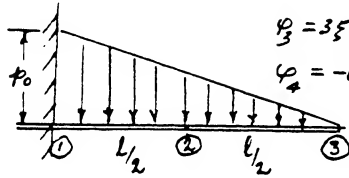
$$\begin{vmatrix} (12 + \frac{kl^3}{EI} - 366\lambda) & (6 - 22\lambda)l \\ (6 - 22\lambda)l & (10 - 5\lambda)l^2 \end{vmatrix} = 0$$

$$\lambda^2 - (2.567 + .003715 \frac{kl^3}{EI}) \lambda + .062407 = 0$$

\therefore must decide on numerical value of $\frac{kl^3}{EI}$ before solving

If $\frac{kl^3}{EI} = 0$, then $\lambda_1 = .02454 \quad \omega_1 = 3.21 \sqrt{EI/ml^4}$
 $\lambda_2 = 2.543 \quad \omega_2 = 32.63 \sqrt{EI/ml^4}$

10-28(a)



$$\begin{aligned}\varphi_1 &= 1 - 3\xi^2 + 2\xi^3 \\ \varphi_2 &= l\xi - l\xi^2 + l\xi^3 \\ \varphi_3 &= 3\xi^2 - 2\xi^3 \\ \varphi_4 &= -l\xi^2 + l\xi^3\end{aligned}$$

Element ①-② See Eq. (10.7-4)

Since $\bar{U}_1 = 0$ only φ_3 & φ_4 are needed.

$$\bar{F}_2 = \int_0^{l_1} p_0 \left(1 - \frac{x}{l_1}\right) (3\xi^2 - 2\xi^3) l_1 d\xi$$

$$M_2 = \int_0^{l_1} p_0 \left(1 - \frac{x}{l_1}\right) (-l\xi^2 + l\xi^3) l_1 d\xi$$

Integrated results.

$$\bar{F}_2 = .3250 p_0 l_1$$

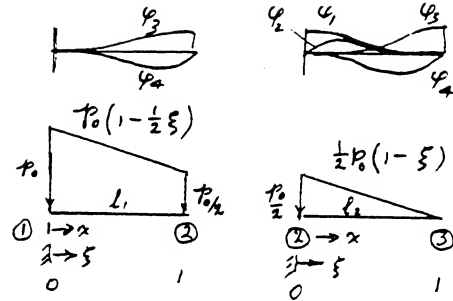
$$M_2 = -.0583 p_0 l_1^2$$

Assembly of both sections

$$\begin{Bmatrix} \bar{F}_2 \\ M_2 \\ \bar{F}_3 \\ M_3 \end{Bmatrix} = -p_0 \begin{Bmatrix} .325 l_1 + .250 l_2 \\ -.0583 l_1^2 - .850 l_2^2 \\ .0750 l_2 \\ .0583 l_2^2 \end{Bmatrix}$$

with $l_1 = l_2 = l/2$

$$\begin{Bmatrix} \bar{F}_2 \\ M_2 \\ \bar{F}_3 \\ M_3 \end{Bmatrix} = -p_0 l \begin{Bmatrix} .2875 \\ -.2271 \\ .0375 \\ .0146 \end{Bmatrix}$$



Element ②-③ See Eq. (10.7-4)

all 4 φ_i are necessary

$$\bar{F}_3 = \int_0^{l_2} \frac{p_0}{2} (1 - \xi) (3\xi^2 - 2\xi^3) l_2 d\xi$$

$$M_3 = \int_0^{l_2} \frac{p_0}{2} (1 - \xi) (-l_2 \xi^2 + l_2 \xi^3) l_2 d\xi$$

$$\bar{F}_2 = -\int_0^{l_2} \frac{p_0}{2} (1 - \xi) (1 - 3\xi^2 + 2\xi^3) l_2 d\xi$$

$$M_2 = -\int_0^{l_2} \frac{p_0}{2} (1 - \xi) (l_2 \xi - 2l_2 \xi^2 + l_2 \xi^3) l_2 d\xi$$

Integrated results

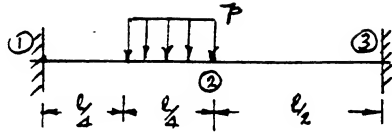
$$\bar{F}_2 = .25 p_0 l_2$$

$$M_2 = -.850 p_0 l_2^2$$

$$\bar{F}_3 = .0750 p_0 l_2$$

$$M_3 = .0583 p_0 l_2^2$$

10-28(b)



$$\begin{aligned}\bar{F}_2 &= -p \int_{l/2}^l \varphi_3 l d\xi = -p \int_{l/2}^l (3\xi^2 - 2\xi^3) l d\xi = -pl \left[3\left(\frac{\xi^3}{3}\right) - 2\left(\frac{\xi^4}{4}\right) \right]_{l/2}^l \\ &= -p \left[.4063 \right] l, = -pl \left[.2031 \right] \quad l_1 = \frac{l}{2}\end{aligned}$$

$$M_2 = -p \int_{l/2}^l (-l_1 \xi^2 + l_1 \xi^3) l d\xi = .0286 pl^2$$

with $\bar{u}_1 = \bar{u}_3 = \theta_1 = \theta_3 = 0$ (see Ex. 10.6-1)

$$\begin{Bmatrix} -P_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 192 & \\ & 16l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_2 \\ \theta_2 \end{Bmatrix} = -pl \begin{Bmatrix} .2031 \\ .0286l \end{Bmatrix}$$

$$\therefore \bar{u}_2 = - \frac{.2031}{192} \frac{pl^4}{EI}$$

$$\theta_2 = - \frac{.0286}{16} \frac{pl^3}{EI}$$

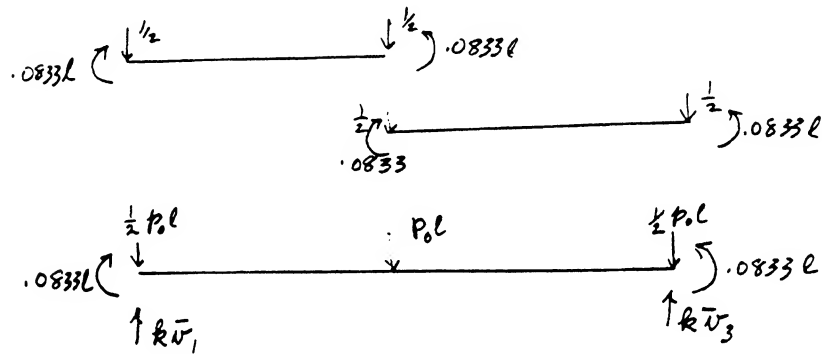
10-29 Cont.

$$\bar{F}_1 = -p_0 f(t) \int_0^1 \varphi_1 l d\xi = -p_0 f(t) \left[1 - 1 + \frac{1}{2} \right] l = -\frac{1}{2} p_0 l f(t)$$

$$\bar{M}_1 = -p_0 f(t) l \int_0^1 \varphi_2 l d\xi = -p_0 l^2 f(t) \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = -.0833 p_0 l^2 f(t)$$

$$\bar{F}_2 = -p_0 f(t) \int_0^1 \varphi_3 l d\xi = -p_0 l f(t) \left[1 - \frac{2}{4} \right] = -\frac{1}{2} p_0 l f(t)$$

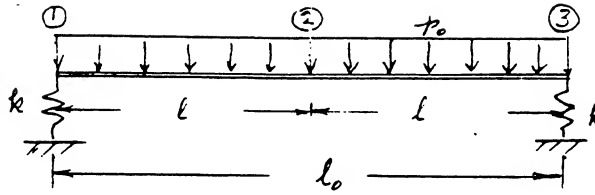
$$\bar{M}_2 = -p_0 f(t) l \int_0^1 \varphi_4 l d\xi = -p_0 l^2 f(t) \left[-\frac{1}{3} + \frac{1}{4} \right] = .0833 p_0 l^2 f(t)$$



$$\begin{Bmatrix} \bar{F}_1 - k\bar{v}_1 \\ \bar{M}_1 \\ \bar{F}_2 \\ \bar{M}_2 \\ \bar{F}_3 - k\bar{v}_3 \\ \bar{M}_3 \end{Bmatrix} = \begin{Bmatrix} -.50 - k\bar{v}_1 \\ -.0833 l \\ -1.0 \\ 0 \\ -.50 - k\bar{v}_3 \\ .0833 l \end{Bmatrix} p_0 l f(t)$$

$-k\bar{v}_1$ and $-k\bar{v}_3$ must now be shifted to stiffness matrix as $+k\bar{v}_1$ and $+k\bar{v}_3$

10-29



Element (1-2) Eq.(10.2-1) for \bar{k} and Eq.(10.2-10) for \bar{m}

$$\bar{k} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \quad \bar{m} = \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Assembled

$$\bar{k} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{bmatrix} \bar{N}_1 \\ \theta_1 \\ \bar{N}_2 \\ \theta_2 \\ \bar{N}_3 \\ \theta_3 \end{bmatrix}$$

must replace l by $\frac{l_0}{2}$

$$\bar{m} = \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

For loads, must determine generalized forces & moments at stations

10-29 Cont.

General equation of motion 6x6

$$\frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l & 0 & 0 \\ 22l & 4l^2 & 13l & -3l^2 & 0 & 0 \\ 54 & 13l & 312 & 0 & 54 & -13l \\ -13l & -3l & 0 & 8l^2 & 13l & -3l^2 \\ 0 & 0 & 54 & 13l & 156 & -22l \\ 0 & 0 & -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{\theta}_1 \\ \ddot{u}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{EI}{l^3} \begin{bmatrix} (12+\frac{kl^3}{EI}) & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -1l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & (12+\frac{kl^3}{EI}) & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \theta_1 \\ \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -50 \\ -0.0336 \\ -1.0 \\ 0 \\ -50 \\ 0.0336 \end{Bmatrix} \quad \text{Polynomial}$$

10-30

Symmetric mode (free-free vibration)

$$\bar{u}_1 = \bar{u}_3, \quad \theta_2 = 0, \quad \theta_1 = -\theta_3$$

Due to symmetry only \bar{u}_1 , θ_1 , and \bar{u}_2 are necessary

From above equation we have

$$\frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 \\ 22l & 4l^2 & 13l \\ 54 & 13l & 312 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{\theta}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{l^3} \begin{bmatrix} (12+\frac{kl^3}{EI}) & 6l & -12 \\ 6l & 4l^2 & -6l \\ -12 & -6l & 24 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \theta_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \frac{kl^3}{EI} = 2$$

Computer solution follows:

With $\frac{kl^3}{EI} = 2$, the end springs are very flexible and the beam behaves almost like a rigid bar, as shown by the first mode

10-30 cont

m =

156	22	54
22	4	13
54	13	312

k =

14	6	-12
6	4	-6
-12	-6	24

>>a=qin'*k*qin

$q_{in} = \text{inv}(\text{chol}(m))$

a =

0.0897	0.3402	-0.2032
0.3402	2.8817	-1.3864
-0.2032	-1.3864	0.7490

>>[v,d]=eig(a)

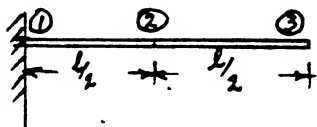
v =

0.8052	-0.1116	-0.5824
0.1784	-0.8911	0.4173
0.5656	0.4399	0.6976

d =

0.0224	0	0
0	3.6087	0
0	0	0.0893

10-3 | Eq. of motion for Example 10.5-1 is

$$\begin{bmatrix} -\lambda \begin{bmatrix} 3/2 & 0 & 54 & -6.5l \\ 0 & 2l^2 & 6.5l & -7.5l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -7.5l^2 & -11l & l^2 \end{bmatrix} + \begin{bmatrix} 24 & 0 & -12 & 3l \\ 0 & 2l^2 & -3l & 1.5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & 1.5l & -3l & l^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
 \end{bmatrix}$$


$$\lambda = \left(\frac{\omega^2 m l}{840} \right) \left(\frac{l^3}{8EI} \right) = \frac{\omega^2 m l^4}{6720 EI} \quad \text{where } \frac{l}{2} \text{ has been substituted for } l.$$

Since l inside of matrices factor out of the equation
 Let $l = 1.0$ inside of the matrices

Input into computer program with $l = 1$.

Solution gives

$$\omega_1 = 3.517 \sqrt{\frac{EI}{ml^4}}$$

$$\omega_2 = 22.22 \quad "$$

$$\omega_3 = 75.16 \quad "$$

$$\omega_4 = 218.1 \quad "$$

10-31 cont

≡

m =

312.0000	0	54.0000	-6.000
0	2.0000	6.5000	-0.7500
54.0000	6.5000	156.0000	-11.0000
-6.5000	-0.7500	-11.0000	1.0000

»k

k =

24.0000	0	-12.0000	3.0000
0	2.0000	-3.0000	0.5000
-12.0000	-3.0000	12.0000	-3.0000
3.0000	0.5000	-3.0000	1.0000

a =

0.0769	0	-0.0816	0.3811
0	1.0000	-0.5996	1.2864
-0.0816	-0.5996	0.4580	-1.3319
0.3811	1.2864	-1.3319	6.4619

»[v,d]=eig(a)

v =

-0.0542	0.1011	0.5904	-0.7989
-0.2219	-0.8784	0.3807	0.1852
0.2119	0.3722	0.7108	0.5580
-0.9502	0.2823	0.0359	0.1267

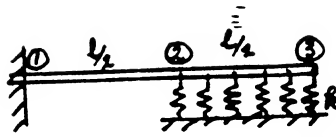
$$\lambda = \frac{\omega^2 m l^4}{6720 EI}$$

d =

7.0810	0	0	0
0	0.8406	0	0
0	0	0.0018	0
0	0	0	0.0735

$$\omega = \begin{matrix} 218.13 \\ 75.16 \\ 3.517 \\ 22.22 \end{matrix}$$

10-32
See Ex. 10.8-1



For 2 element beam, the form of the equation of motion is

$$\frac{ml}{840} [m_{ij}] \begin{Bmatrix} \ddot{u}_2 \\ \ddot{\theta}_2 \\ \ddot{u}_3 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{8EI}{l^3} [k_{ij}] \begin{Bmatrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} + \frac{8EI}{l^3} \left(\frac{kl^3}{8EI} \right) [q_{ij}] \begin{Bmatrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $\{Q_i\}$ in text is modified for $l = \frac{l}{2}$ and $\frac{kl^3}{8EI} = 10$.

$$\{Q_i\} = -k \begin{bmatrix} .1857l & .0131l^2 & .0643l & -.007736l^2 \\ & .001191l^3 & .007738l^2 & -.000893l^3 \\ & & .1857l & -.013075l^2 \\ & & & .001191l^3 \end{bmatrix} \times 10.$$

$$\text{with } \lambda = \frac{\omega^2 ml^4}{6720 EI}$$

$$\left[-\lambda [m_{ij}] + \left[k_{ij} + \frac{kl^3}{8EI} q_{ij} \right] \right] \begin{Bmatrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$m_{ij} = \begin{bmatrix} 312 & 0 & 54 & -6.5l \\ 0 & 2l^2 & 6.5l & -.75l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -.75l^2 & -11l & l^2 \end{bmatrix}$$

$$k_{ij} = \begin{bmatrix} 24 & 0 & -12 & 3l \\ 0 & 2l^2 & -3l & .5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & .5l^2 & -3l & l^2 \end{bmatrix} \text{ without elastic foundation}$$

add to k_{ij} , $Q_{ij} = + Q_{ji} =$ above eq.

10-32 Cont

»m

m =

312.0000	0	54.0000	-6.5000
0	2.0000	6.5000	-0.7500
54.0000	6.5000	156.0000	-11.0000
-6.5000	-0.7500	-11.0000	1.0000

»k

k =

25.8600	0.1310	-11.3600	2.9220
0.1310	2.0120	-2.9230	0.4910
-11.3600	-2.9230	13.8600	-3.1310
2.9220	0.4910	-3.1310	1.0120

a =

0.0829	0.0052	-0.0822	0.3817
0.0052	1.0060	-0.5986	1.2847
-0.0822	-0.5986	0.4697	-1.3314
0.3817	1.2847	-1.3314	6.4735

»[v,d]=eig(a)

v =

-0.0544	0.0949	0.6111	-0.7839
-0.2215	-0.8779	0.3740	0.2007
0.2118	0.3746	0.6968	0.5739
-0.9503	0.2827	0.0331	0.1260

d =

7.0915	0	0	0
0	0.8472	0	0
0	0	0.0131	0
0	0	0	0.0804

$$\lambda = \omega^2 m l^4$$

$$4720 \text{ EI}$$

$$\omega = 218$$

$$75.45$$

$$\sqrt{\frac{\text{EI}}{m l^4}}$$

$$23.23$$

$$9.378$$

10-33

Same as Prob 10-32 but with pin at ①

$$\therefore V_1 = 0, \quad \theta_1 \neq 0$$

$$K = \frac{8EI}{l^3} \begin{bmatrix} 1 & -3 & .5 & 0 & 0 \\ -3 & 24 & 0 & -12 & 3 \\ .5 & 0 & 2 & -3 & .5 \\ 0 & -12 & -3 & 12 & -3 \\ 0 & 3 & .5 & -3 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{Bmatrix}$$

with $l=1$ inside matrix

$$M = \frac{ml}{840} \begin{bmatrix} 1 & 6.5 & -.75 & 0 & 0 \\ 6.5 & 312 & 0 & 54 & -6.5 \\ -.75 & 0 & 2 & 6.5 & -.75 \\ 0 & 54 & 6.5 & 156 & -11 \\ 0 & -6.5 & -.75 & -11 & 1 \end{bmatrix} \text{ with } l=1, \text{ inside matrix}$$

$$\begin{Bmatrix} Q_i \end{Bmatrix} = - \left(\frac{kl^3}{8EI} \right) \left(\frac{8EI}{l^3} \right) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & .1857 & .0131 & .0643 & -.007736 \\ 0 & .0131 & .001191 & .007738 & -.000893 \\ 0 & .0643 & .007738 & .1857 & -.013075 \\ 0 & -.00888 & -.000893 & -.013075 & .001191 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \\ \theta_3 \end{Bmatrix}$$

New Stiffness with elastic foundation with $\frac{kl^3}{8EI} = 10$

$$\frac{8EI}{l^3} \begin{bmatrix} 1 & -3 & .5 & 0 & 0 \\ -3 & 25.86 & .131 & -11.357 & 2.923 \\ .5 & .131 & 2.0119 & -2.923 & -.491 \\ 0 & -11.357 & -2.923 & 13.857 & -3.1307 \\ 0 & 2.923 & .491 & -3.1307 & 1.0119 \end{bmatrix}$$

0-33 CONT.

»m

m =

1.0000	6.5500	-0.7500	0	0
6.5000	312.0000	0	54.0000	-6.5000
-0.7500	0	2.0000	6.5000	-0.7500
0	54.0000	6.5000	156.0000	-11.0000
0	-6.5000	-0.7500	-11.0000	1.0000

»k

k =

1.0000	-3.0000	0.5000	0	0
-3.0000	25.8600	0.1310	-11.3600	2.9230
0.5000	0.1310	2.0120	-2.9230	0.4910
0	-11.3600	-2.9230	13.8600	-3.1310
0	2.9230	0.4910	-3.1310	1.0120

a =

1.0000	-0.5822	1.2269	-0.3532	0.5713
-0.5822	0.4016	-0.6448	0.0946	0.1294
1.2269	-0.6448	2.7725	-1.2088	2.3238
-0.3532	0.0946	-1.2088	0.7127	-1.7975
0.5713	0.1294	2.3238	-1.7975	7.6275

»[v,d]=eig(a)

v =

0.1311	-0.4785	0.6513	0.5105	-0.2625
-0.0268	0.3343	-0.3957	0.4519	-0.7258
0.3888	-0.6818	-0.4418	-0.3241	-0.2893
-0.2458	0.1319	0.4262	-0.6481	-0.5661
0.8778	0.4206	0.2057	-0.1004	-0.0133

d =

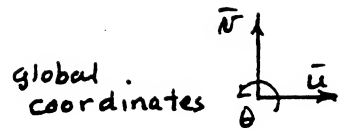
9.2416	0	0	0	0
0	2.7500	0	0	0
0	0	0.4707	0	0
0	0	0	0.0418	0
0	0	0	0	0.0101

$$\lambda = \frac{\omega^2 m l^4}{6720 EI}$$

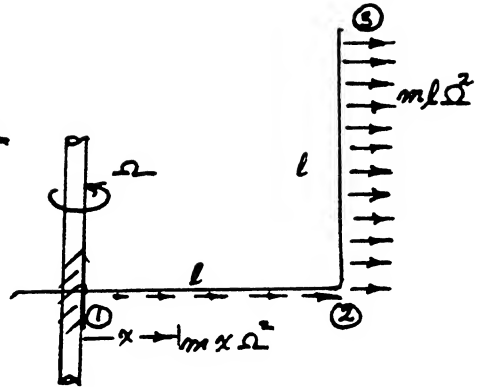
$$\omega = \begin{matrix} 249 \\ 135.6 \\ 56.21 \\ 16.76 \\ 8.267 \end{matrix}$$

$$\sqrt{\frac{EI}{m l^4}}$$

10-34



Eq. 10.4-7 ~ K
Eq. 10.4-8 ~ M



Element ①-②

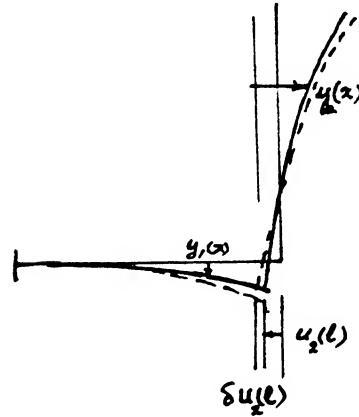
$$\text{Load} = \Omega^2 m x dx$$

$$\text{virtual displ} = \int_0^x y_i' \delta y_i' dx$$

$$= -\sum_i \sum_j q_i \delta q_j \int_0^x \phi_i' \phi_j' dx$$

$$\delta W = -\Omega^2 m \sum_i \sum_j q_i \delta q_j \int_0^l x \int_0^x \phi_i' \phi_j' dx dx$$

$$Q_i = -\Omega^2 m \sum_j q_j \int_0^l x \int_0^x \phi_i' \phi_j' dx dx$$



Element ②-③

$$\text{Load} = \Omega^2 l, m dx$$

$$\text{virtual displ.} = \delta y_2(x) - \delta u_2(l)$$

$$\delta W = \Omega^2 l, m \left[\int_0^l \delta y_2(x) dx - \int_0^l y_i' \delta y_i' dx \right]$$

$$= \Omega^2 l, m \left[\sum_j \delta q_j \int_0^l \phi_j dx - \sum_i \sum_j q_j \delta q_i \int_0^l \phi_i' \phi_j' dx \right]$$

$$Q_i = \frac{\delta W}{\delta q_i} = \Omega^2 l, m \left[\sum_j \int_0^l \phi_j dx - \sum_j q_j \int_0^l \phi_i' \phi_j' dx \right]$$

See Ex 10.8-2 for evaluation of integrals

$y(x) = \phi_1(x) \psi_1 + \phi_2(x) \theta_1 + \phi_3(x) \psi_2 + \phi_4(x) \theta_2$ for any element betw sta. 1 and 2.

10-34 cont.

Stiffness without rotation, $\Omega = 0$ — Eq. 10.4-7

$$\alpha = 0$$

$$C = 1$$

$$S = 0$$

$$K_{12} = \frac{EI}{l^3} \left[\begin{array}{ccc|ccc} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & 0 & 12 & -6 \\ \dots & \dots & \dots & 0 & -6 & 4 \end{array} \right] \left\{ \begin{array}{c} \dots \\ \bar{u}_2 \\ \theta_2 \end{array} \right\}$$

$$\alpha = 90^\circ$$

$$C = 0$$

$$S = 1$$

$$K_{23} = \frac{EI}{l^3} \left[\begin{array}{ccc|ccc} x & x & x & x & x & x \\ x & R & 0 & 0 & -R & 0 \\ x & 0 & 4 & 6 & 0 & 2 \\ x & 0 & 6 & 12 & 0 & 6 \\ x & -R & 0 & 0 & R & 0 \\ x & 0 & 2 & 6 & 0 & 4 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_2 = 0 \\ \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{u}_3 \\ \theta_3 \end{array} \right\}$$

Assembled

since $\bar{u}_2 = \bar{u}_3$ add columns 2 and 5 and delete row 5

$$K = \frac{EI}{l^3} \left[\begin{array}{ccc|ccc} x & x & x & x & x & x \\ x & (12+R) & -6 & 0 & -R & 0 \\ x & -6 & 8 & 6 & 0 & 2 \\ x & 0 & 6 & 12 & 0 & 6 \\ x & -R & 0 & 0 & R & 0 \\ x & 0 & 2 & 6 & 0 & 4 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{u}_3 \\ \theta_3 \end{array} \right\} = \frac{EI}{l^3} \left[\begin{array}{ccc|ccc} -12 & -6 & 0 & 0 & 0 & 0 \\ -6 & 8 & 6 & 2 & 0 & 0 \\ 0 & 6 & 12 & 6 & 0 & 0 \\ 0 & 2 & 6 & 4 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{array} \right\}$$

Mass matrix Eq. 10.4-8

$$\alpha = 0$$

$$C = 1$$

$$S = 0$$

$$M_{12} = \frac{ml}{420} \left[\begin{array}{ccc|ccc} x & x & x & x & x & x \\ x & 156 & 22 & 0 & 54 & -13 \\ x & 22 & 4 & 0 & 13 & -3 \\ x & 0 & 0 & 0 & 0 & 0 \\ x & 54 & 13 & 0 & 156 & -22 \\ x & -13 & -3 & 0 & -22 & 4 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_1 = 0 \\ \bar{u}_1 \\ \theta_1 = 0 \\ \theta_1 \\ \bar{u}_2 \\ \theta_2 \end{array} \right\}$$

$$\alpha = 90$$

$$C = 0$$

$$S = 1$$

$$M_{23} = \frac{ml}{420} \left[\begin{array}{ccc|ccc} x & x & x & x & x & x \\ x & N & 0 & 0 & \frac{1}{2}N & 0 \\ x & 0 & 4 & -13 & 0 & -3 \\ x & 0 & -13 & 156 & 0 & 22 \\ x & \frac{1}{2}N & 0 & 0 & N & 0 \\ x & 0 & -3 & 22 & 0 & 4 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \bar{u}_3 \\ \bar{u}_3 \\ \theta_3 \end{array} \right\}$$

with $\bar{u}_2 = \bar{u}_3$ add columns 2 & 5 & delete row 5

$$M = \frac{ml}{420} \left[\begin{array}{ccc|ccc} 366 & -22 & 0 & 0 & 0 & 0 \\ -22 & 8 & -13 & -3 & 0 & 0 \\ 0 & -13 & 156 & 22 & 0 & 0 \\ 0 & -3 & 22 & 4 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \bar{u}_2 \\ \theta_2 \\ \bar{u}_3 \\ \theta_3 \end{array} \right\}$$

10-34 CONT.

$$\underline{u} = 0$$

»m=[366 -22 0 0 ; -22 8 -13 -3; 0 -13 156 22; 0 -3 22 4]

m =

366	-22	0	0
-22	8	-13	-3
0	-13	156	22
0	-3	22	4

»k=[12 -6 0 0 ; -6 8 6 2 ; 0 6 12 6 ; 0 2 6 4]

k =

12	-6	0	0
-6	8	6	2
0	6	12	6
0	2	6	4

a =

0.0328	-0.1068	-0.0470	-0.0711
-0.1068	1.0965	0.6856	1.3326
-0.0470	0.6856	0.4829	1.2102
-0.0711	1.3326	1.2102	5.4162

»[v,d]=eig(a)

v =

0.5728	0.8135	0.0993	-0.0176
0.4220	-0.1883	-0.8417	0.2792
-0.7002	0.5464	-0.3957	0.2340
0.0602	-0.0654	0.3537	0.9311

d =

$$\lambda = \frac{\omega^2 m l^4}{420 EI}$$

0.0041	0	0	0
0	0.0317	0	0
0	0	0.8714	0
0	0	0	6.1213

$$\omega = \begin{matrix} 1.2809 \\ 3.647 \\ 19.13 \\ 50.70 \end{matrix}$$

10-34 cont

Generalized force Q_i

Element 1-2 $-m\Omega^2 l$ $\left[\begin{array}{c|cc} & & \\ \hline & & \\ \hline & .42857 & -.064286 \\ & -.064286 & .023810 \end{array} \right] \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$

Element 2-3 $m\Omega^2 l \int_0^l \phi_j dx = \Omega^2 ml \begin{Bmatrix} .50l \\ .0833l^2 \\ .50l \\ -.0833l^2 \end{Bmatrix}$ must remain on right side of eq. as force & moment

$\delta u_1(l) = -m\Omega^2 l \sum_j q_j \int_0^l \phi_i' \phi_j' dx = -m\Omega^2 l \begin{Bmatrix} 1.2 & .1l & -1.2 & .1l \\ & .133l^2 & -.1l & -.0333l^2 \\ & & 1.2 & -.1l \\ & & & .133l^2 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix}$

Total Stiffness

$\frac{EI}{l^3} \begin{Bmatrix} 12 & -6 & 0 & 0 \\ -6 & 8 & 6 & 2 \\ 0 & 6 & 12 & 6 \\ 0 & 2 & 6 & 4 \end{Bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} + m\Omega^2 l \begin{Bmatrix} & & & \\ & & & \\ & .42853 & -.06428 & \\ & -.06428 & .02381 & \end{Bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$
 $+ m\Omega^2 l \begin{Bmatrix} & & & \\ & & & \\ & 1.2 & -.1 & \\ & -.1 & .1333 & \end{Bmatrix} \begin{Bmatrix} \bar{v}_2 \\ \theta_2 \end{Bmatrix}$ To combine with $\frac{EI}{l^3}$ multiply by $\frac{l^3}{EI}$ to obtain $(\frac{m\Omega^2 l^4}{EI})$ chosen as 1.0

$= \frac{EI}{l^3} \begin{Bmatrix} 13.628 & -6.16428 & 0 & 0 \\ -6.16428 & 8.15714 & 6 & 2 \\ 0 & 6 & 12 & 6 \\ 0 & 2 & 6 & 4 \end{Bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \bar{v}_3 \\ \theta_3 \end{Bmatrix}$ with

Input into Choljac. with Ω , ie $\frac{m\Omega^2 l^4}{EI} = 1$, $\lambda = \frac{\omega^2 ml^4}{420 EI}$

10.34 Cont.

$$\frac{m \omega^2 l^4}{EI} = 1.0$$

m matrix as previous case

```
>>k=[13.63 -6.164 0 0 ; -6.164 8.157 6 2 ; 0 6 12 6 ; 0 2 6 4]
```

k =

```
13.6300    -6.1640         0         0
-6.1640     8.1570     6.0000     2.0000
         0     6.0000    12.0000     6.0000
         0     2.0000     6.0000     4.0000
```

```
>>a=qin'*k*qin
```

a =

```
0.0372    -0.1081    -0.0476    -0.0720
-0.1081     1.1180     0.6951     1.3469
-0.0476     0.6951     0.4871     1.2165
-0.0720     1.3469     1.2165     5.4257
```

```
>>[v,d]=eig(a)
```

v =

```
-0.0178     0.0992    -0.5175     0.8497
0.2821    -0.8419    -0.4318    -0.1588
0.2349    -0.3928     0.7359     0.4989
0.9300     0.3565    -0.0648    -0.0616
```

$$\lambda = \frac{\omega^2 m l^4}{420 EI}$$

d =

```
6.1429         0         0         0
         0     0.8846         0         0
         0         0     0.0057         0
         0         0         0     0.0347
```

$$\omega = \begin{matrix} 50.8 \\ 19.276 \\ 3.8187 \\ 1.544 \end{matrix}$$

10-35

From Eq. 10.2-1 and Sec. 10.6 the stiffness for the 1st element of length $\ell/2$ with spring K at the pinned end is

$$\frac{8EI}{\ell^3} \begin{bmatrix} 4(\frac{\ell}{2})^2 & -6(\frac{\ell}{2}) & 2(\frac{\ell}{2})^2 \\ -6(\frac{\ell}{2}) & 12 & -6(\frac{\ell}{2}) \\ 2(\frac{\ell}{2})^2 & -6(\frac{\ell}{2}) & 4(\frac{\ell}{2})^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} + K \theta_1$$

$$= \frac{8EI}{\ell^3} \begin{bmatrix} (\ell^2 + \frac{K\ell^3}{8EI}) & -3\ell & .5\ell^2 \\ -3\ell & 12 & -3\ell \\ .5\ell^2 & -3\ell & \ell^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

When rotated, the generalized force given in Ex. 10.8-2 for element of length ℓ , requires adding the following term

$$m\Omega^2 \frac{\ell}{2} \begin{bmatrix} .05714(\frac{\ell}{2})^2 & -.01429(\frac{\ell}{2}) & -.009524(\frac{\ell}{2})^2 \\ & .4286 & -.06429(\frac{\ell}{2}) \\ & & .02381(\frac{\ell}{2})^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

10-36 10-37 Discussed in Example 10.8-4 (text)
10-38 let $l = 1.0$ inside matrices, $u_1 = \theta_1 = 0$

$$K_{1-2} = \frac{27EI}{l^3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & -6 & 0 \\ 0 & -6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{matrix}$$

$$K_{2-3} = \frac{27EI}{l^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{matrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{matrix}$$

$$K_{3-4} = \frac{27EI}{l^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{matrix} u_3 \\ \theta_3 \\ u_4 \\ \theta_4 \end{matrix}$$

Assemble

$$K = \frac{27EI}{l^3} \begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{matrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \\ u_4 \\ \theta_4 \end{matrix}$$

$$M = \frac{ml}{420 \times 3} \begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix} \begin{matrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \\ u_4 \\ \theta_4 \end{matrix}$$

The above 6×6 matrices are inputted into Choljac
 with $\lambda = \frac{\omega^2 ml^4}{420 \times 3 \times 27EI} = \frac{\omega^2 ml^4}{34020EI}$

To obtain reduced stiffness, see Ex 10.5-2 (text)

1st rearrange rows with displacements in first three rows
 and angles in rows 4, 5, 6.

Then rearrange columns to conform to new column vector

10-58 Cont.

For stiffness matrix

Rearranged rows

$$\begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

Rearrange columns

$$\begin{bmatrix} 24 & -12 & 0 & 0 & 6 & 0 \\ -12 & 24 & -12 & -6 & 0 & 6 \\ 0 & -12 & 12 & 0 & -6 & -6 \\ 0 & -6 & 0 & 8 & 2 & 0 \\ 6 & 0 & -6 & 2 & 8 & 2 \\ 0 & 6 & -6 & 0 & 2 & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

$$= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} V \\ \Theta \end{Bmatrix}$$

For the mass Matrix

The rearranged mass matrix is

$$\begin{bmatrix} 312 & 54 & 0 & 0 & -13 & 0 \\ 54 & 312 & 54 & 13 & 0 & -13 \\ 0 & 54 & 156 & 0 & 13 & 22 \\ 0 & 13 & 0 & 8 & -3 & 0 \\ -13 & 0 & 13 & -3 & 8 & -3 \\ 0 & -13 & -22 & 0 & -3 & 4 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} V \\ \Theta \end{Bmatrix}$$

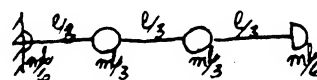
Assume $M_{12} = M_{21} = M_{22} = 0$

$$\begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{V} \\ \ddot{\Theta} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} V \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$M_{11} \ddot{V} + K^* V = 0 \quad = 3 \times 3 \text{ equation}$$

where $K^* = K_{11} - K_{12} K_{22}^{-1} K_{21}$ = reduced stiffness

Although $M_{11} = \frac{m l}{420 \times 3} \begin{bmatrix} 312 & 54 & 0 \\ 54 & 312 & 54 \\ 0 & 54 & 156 \end{bmatrix}$ we can choose discrete masses as follows



10-38 cont.

ie. assume half the mass of each element at the ends

$$M_{11} = m \cdot l \begin{bmatrix} .333 & 0 & 0 \\ 0 & .333 & 0 \\ 0 & 0 & .1665 \end{bmatrix}$$

For K^*

$$K_{22}^{-1} = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix}^{-1} = \frac{1}{208} \begin{bmatrix} 28 & -8 & 4 \\ -8 & 32 & -16 \\ 4 & -16 & 60 \end{bmatrix}$$

$$K_{12} K_{22}^{-1} K_{21} = \begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & 6 \\ 0 & -6 & -6 \end{bmatrix} \begin{bmatrix} 28 & -8 & 4 \\ -8 & 32 & -16 \\ 4 & -16 & 60 \end{bmatrix} \begin{bmatrix} 0 & -6 & 0 \\ 6 & 0 & -6 \\ 0 & 6 & -6 \end{bmatrix} \frac{1}{208}$$

$$= \begin{bmatrix} 1152 & -288 & -576 \\ -288 & 2880 & -1728 \\ -576 & -1728 & 2160 \end{bmatrix} \frac{1}{208} = \begin{bmatrix} 5.538 & -1.3846 & -2.769 \\ -1.3846 & 13.846 & -8.308 \\ -2.769 & -8.308 & 10.385 \end{bmatrix}$$

$$K^* = \begin{bmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{bmatrix} - \begin{bmatrix} 5.538 & -1.3846 & -2.769 \\ -1.3846 & 13.846 & -8.308 \\ -2.769 & -8.308 & 10.385 \end{bmatrix}$$

$$= \begin{bmatrix} 18.462 & -10.615 & 2.769 \\ -10.615 & 10.154 & -3.692 \\ 2.769 & -3.692 & 1.615 \end{bmatrix} \frac{27EI}{l^3}$$

$$\lambda = (\omega^2 m l) \frac{l^3}{27EI} = \frac{\omega^2}{27} \frac{m l^4}{EI}$$

$$\omega = \sqrt{27\lambda} \sqrt{\frac{EI}{m l^4}}$$

10-38 cont

$$\gg m = [.333 \ 0 \ 0 ; 0 \ .333 \ 0 ; 0 \ 0 \ .1665]$$

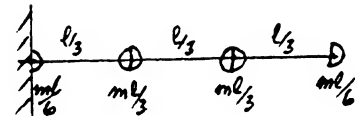
m =

$$\begin{bmatrix} 0.3330 & 0 & 0 \\ 0 & 0.3330 & 0 \\ 0 & 0 & 0.1665 \end{bmatrix}$$

$$\gg k = [18.46 \ -10.61 \ 2.769; -10.61 \ 10.15 \ -3.692; 2.769 \ -3.692 \ 1.615]$$

k =

$$\begin{bmatrix} 18.4600 & -10.6100 & 2.7690 \\ -10.6100 & 10.1500 & -3.6920 \\ 2.7690 & -3.6920 & 1.6150 \end{bmatrix}$$



a =

$$\begin{bmatrix} 55.4354 & -31.8619 & 11.7596 \\ -31.8619 & 30.4805 & -15.6795 \\ 11.7596 & -15.6795 & 9.6997 \end{bmatrix}$$

$$\gg [v, d] = \text{eig}(a)$$

v =

$$\begin{bmatrix} 0.7873 & -0.5901 & 0.1786 \\ -0.5634 & -0.5710 & 0.5971 \\ 0.2503 & 0.5708 & 0.7820 \end{bmatrix}$$

$$\lambda = \frac{\omega^2 m l^4}{27 EI}$$

d =

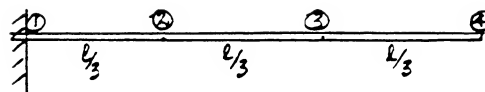
$$\begin{bmatrix} 81.9738 & 0 & 0 \\ 0 & 13.2281 & 0 \\ 0 & 0 & 0.4137 \end{bmatrix}$$

$$\omega = \begin{matrix} 47.05 \\ 18.89 \\ 3.347 \end{matrix} \sqrt{\frac{EI}{m l^4}}$$

10-38 cont.

m =

312	0	54	-13	0	0
0	8	13	-3	0	0
54	13	312	0	54	-13
-13	-3	0	8	13	-3
0	0	54	13	156	-22
0	0	-13	-3	-22	4



k =

24	0	-12	6	0	0
0	8	-6	2	0	0
-12	-6	24	0	-12	6
6	2	0	8	-6	2
0	0	-12	-6	12	-6
0	0	6	2	-6	4

a =

0.0769	0	-0.0545	0.1691	-0.0555	0.0912
0	1.0000	-0.4004	0.7812	-0.2136	0.3415
-0.0545	-0.4004	0.2469	-0.3327	0.0045	0.2650
0.1691	0.7812	-0.3327	1.9215	-0.9207	1.8330
-0.0555	-0.2136	0.0045	-0.9207	0.5897	-1.5889
0.0912	0.3415	0.2650	1.8330	-1.5889	7.1247

»[v,d]=eig(a)

v =

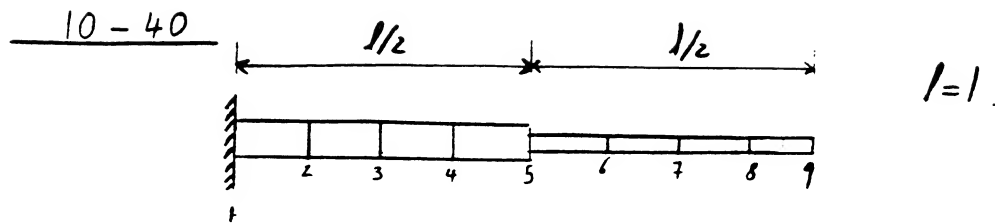
0.0183	-0.0567	-0.1449	-0.6481	0.7052	-0.2409
0.0837	-0.5490	0.7351	0.2340	0.2639	-0.1636
0.0130	0.2971	-0.2094	0.5674	0.2667	-0.6889
0.3126	-0.6894	-0.4922	-0.0446	-0.3072	-0.2974
-0.2321	0.1647	0.3399	-0.4345	-0.5128	-0.5930
0.9170	0.3237	0.1926	-0.1113	-0.0671	-0.0192

$$\lambda = \frac{\omega^2 m l^4}{34020 EI}$$

d =

8.1884	0	0	0	0	0
0	2.0602	0	0	0	0
0	0	0.5817	0	0	0
0	0	0	0.1147	0	0
0	0	0	0	0.0144	0
0	0	0	0	0	0.0004

$$\omega = \begin{matrix} 527.6 \\ 264.5 \\ 140.796 \\ 63.520 \\ 22.11 \\ 3.514 \end{matrix} \sqrt{\frac{EI}{m l^4}}$$



The mass and stiffness matrices for the elements in the first (larger in diameter) section are given as

$$\frac{1}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad 16 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For the other section, they are given as

$$\frac{1}{24} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad 8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$u_1 = 0$. Therefore, the mass and stiffness for the system are given as (neglect u_1)

$$M = \frac{11}{24} \begin{bmatrix} 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 8 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 8 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 8 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 8 \end{bmatrix}, \quad K = 8 \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix}$$

Let $u = [u_2 \ u_3 \ u_4 \ u_5 \ u_6 \ u_7 \ u_8 \ u_9]^T$. Therefore, the equation for the longitudinal vibration is

$$M\ddot{u} + Ku = 0.$$

The natural frequencies are the square roots of the generalized eigenvalues of M and K , i.e., $K - \omega^2 M = 0$. Solving for ω , we get

$$\omega = 19.186, 17.628, 14.040, 11.603, 8.096, 6.049, 1.354, 3.130$$

You can get the same answers from the

program pin-bar.m

enter $m = [2, 2, 2, 2, 1, 1, 1, 1]$

$A = [2, 2, 2, 2, 1, 1, 1, 1]$

For the two values to agree recall that the above values have been multiplied by $\frac{1}{\sqrt{2}}$

10-41 cont output from pin-beam.m

omega =

54.5345	0	0	0	2-element beam
0	18.7893	0	0	
0	0	0.8794	0	
0	0	0	5.5554	

omega =

Columns 1 through 7

538.6989	0	0	0	0	0	0
0	390.6837	0	0	0	0	0
0	0	219.6134	0	0	0	0
0	0	0	297.0578	0	0	0
0	0	0	0	160.7123	0	0
0	0	0	0	0	117.0056	0
0	0	0	0	0	0	75.8829
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 12

0	0	0	0	0	6 - element beam
0	0	0	0	0	
0	0	0	0	0	
0	0	0	0	0	
0	0	0	0	0	
0	0	0	0	0	
0	0	0	0	0	
50.7158	0	0	0	0	
0	30.4202	0	0	0	
0	0	15.4525	0	0	
0	0	0	5.5100	0	
0	0	0	0	0.8790	

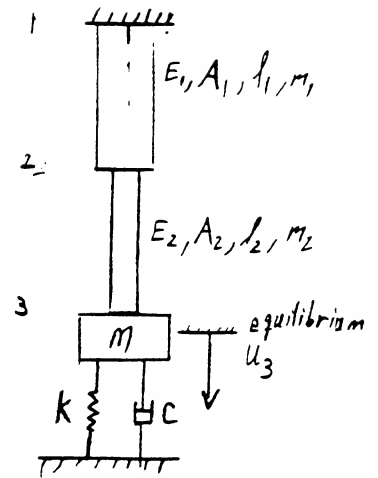
see example 10-5.1

10-42

The mass and stiffness matrices of the system are given as (Ignore u_1 since it is equal to zero) let $u = [u_2 \ u_3]$

$$M = \begin{bmatrix} \frac{m_1 l_1}{3} + \frac{m_2 l_2}{3} & \frac{m_2 l_2}{6} \\ \frac{m_2 l_2}{6} & \frac{m_2 l_2}{3} + m \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} & -\frac{E_2 A_2}{l_2} \\ -\frac{E_2 A_2}{l_2} & \frac{E_2 A_2}{l_2} + k \end{bmatrix}$$



There is also a damping force at 3 equal to $C\dot{u}_3$ opposite to the direction of motion. Therefore, the equation of motion is written as:

$$\frac{1}{3} \begin{bmatrix} m_1 l_1 + m_2 l_2 & m_2 l_2 / 2 \\ m_2 l_2 / 2 & m_2 l_2 + 3m \end{bmatrix} \begin{bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \begin{bmatrix} \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} & -\frac{E_2 A_2}{l_2} \\ -\frac{E_2 A_2}{l_2} & \frac{E_2 A_2}{l_2} + k \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

10-43

For a single beam $v_2 = \theta_2 = M_1 = 0$, we have

$$\begin{bmatrix} F_1 \\ M_1 \end{bmatrix} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l \\ 6l & 4l^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \end{bmatrix}$$

$$\therefore M_1 = 0 \Rightarrow 6l v_1 + 4l^2 \theta_1 = 0 \Rightarrow \theta_1 = -\frac{3}{2l} v_1$$

$$F_1 = \frac{EI}{l^3} [12 v_1 + 6l \theta_1] = \underbrace{\frac{3EI}{l^3}}_{\text{beam stiffness}} v_1 \quad (\text{cantilever})$$

The mass is found by multiplying the first row of the mass matrix by $\begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2 \end{bmatrix}^T$

$$\Rightarrow \frac{ml}{420} [156 \ddot{v}_1 + 22l \ddot{\theta}_1] = \underbrace{\frac{4l}{140}}_{\text{beam mass}} ml \ddot{v}_1$$

Equations of motion

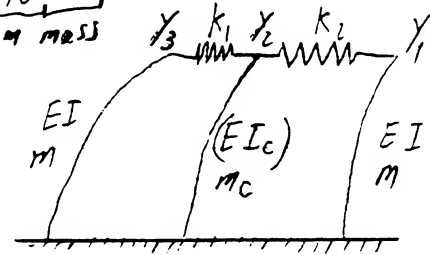
$$1) \frac{4l}{140} ml \ddot{y}_1 = -\frac{3EI}{l^3} y_1 - k_c (y_1 - y_2)$$

$$\Rightarrow \frac{4l}{140} ml \ddot{y}_1 + \left(k_c + \frac{3EI}{l^3} \right) y_1 = k_c y_2$$

$$2) \frac{4l}{140} m_c \ddot{y}_2 = k_c (y_1 - y_2) - k_1 (y_2 - y_3) - \frac{3EI}{l^3} y_2$$

$$\Rightarrow \frac{4l}{140} m_c \ddot{y}_2 + \left(k_1 + k_c + \frac{3EI}{l^3} \right) y_2 = k_1 y_3 + k_c y_1$$

$$3) \frac{4l}{140} m \ddot{y}_3 + \left(k_1 + \frac{3EI}{l^3} \right) y_3 = k_1 y_2$$



If $y_1 = y_2 = y_3$ the forces from k_1 and k_2 are zero.
Free vibration for the three beams

\Rightarrow They must vibrate with the same frequency ω

$$\omega = \sqrt{\frac{3EI/l^3}{\frac{4l}{140} ml}} = \sqrt{\frac{420EI}{4lmg^4}} = \sqrt{\frac{420(EI)_c}{4l m_c l^4}}$$

condition: \therefore Impossible if $\frac{(EI)_c}{m_c} \neq \frac{EI}{m}$

$$11-1 \quad m \ddot{x} + kx = P_0$$

$$x = \frac{P_0}{k} (1 - \cos \omega t), \quad \omega = \sqrt{\frac{k}{m}}$$

$$\therefore x_{\max} = 2 \frac{P_0}{k}$$

11-2

$$\ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = \frac{P_0}{m}$$

$$q_i = \frac{P_0}{m} \left[1 - \frac{e^{-\zeta_i \omega_i t}}{\sqrt{1-\zeta_i^2}} \cos(\sqrt{1-\zeta_i^2} \omega_i t - \psi_i) \right]$$

$$\tan \psi_i = \frac{\zeta_i}{\sqrt{1-\zeta_i^2}} \quad \text{For small damping } \psi_i \approx 0$$

$$\sqrt{1-\zeta_i^2} \approx 1$$

$$\therefore q_i \approx \frac{P_0}{k} [1 - e^{-\zeta_i \omega_i t} \cos \omega_i t] = \frac{P_0}{k} D_i(t)$$

11-3

$$\ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = \frac{f(t)}{M} \int_0^l \frac{P_0}{l} \phi_i(x) dx$$

$$\text{where } f(x, t) = \frac{P_0}{l} f(t) \quad \text{and } P_0 = \text{total force}$$

$$\text{Mode participation factor} = \frac{1}{l} \int_0^l \phi_i(x) dx$$

11-4

$$\text{Generalized force} = \frac{P_0}{M_i} \left\{ \frac{1}{l} \int_0^l p(x) \phi_i(x) dx \right\} f(t)$$

(See Sec. 11.1)

$$K_i = \frac{1}{l} \int_0^l p(x) \phi_i(x) dx, \quad p(x) = l \delta(x-a)$$

$$\therefore K_i = \frac{1}{l} \int_0^l l \delta(x-a) \phi_i(x) dx = \phi_i(a)$$

$$q_i = \frac{P_0 K_i}{M_i \omega_i^2} D_i(t), \quad y(x, t) = \sum_i q_i \phi_i = \sum_i \frac{P_0 \phi_i(a) \phi_i(x)}{M_i \omega_i^2} D_i(t)$$

$$\omega_i^2 = (\beta_i l)^2 \frac{EI}{M_i l^3} \quad \therefore y(x, t) = \frac{P_0 l^3}{EI} \sum_i \frac{\phi_i(a) \phi_i(x)}{(\beta_i l)^4} D_i(t)$$

$$\begin{aligned}
 \underline{11-5} \quad K_i &= \lim_{\epsilon \rightarrow 0} \frac{1}{\ell} \int_0^{\ell} \ell^2 \left[\frac{\delta(x-a-\epsilon) - \delta(x-a)}{\epsilon} \right] \phi_i(x) dx \\
 &= \lim_{\epsilon \rightarrow 0} \ell \left[\frac{\phi_i(a+\epsilon) - \phi_i(a)}{\epsilon} \right] = \ell \left. \frac{d\phi_i(x)}{dx} \right|_{x=a} \\
 &= \beta_i \ell \cdot \frac{1}{\beta_i} \left. \frac{d\phi_i(x)}{dx} \right|_{x=a} = \beta_i \ell \phi_i'(a)
 \end{aligned}$$

See Append. D for ϕ'

$$\underline{11-6} \quad K_i = \frac{1}{\ell} \int_0^{\ell} \ell \delta(x - \frac{\ell}{2}) \phi_i(x) dx = \phi_i(\frac{\ell}{2})$$

$$\phi_n(x) = \sqrt{2} \sin n\pi \frac{x}{\ell} \quad \text{when normalized to total mass } M$$

$$\phi_n(\frac{\ell}{2}) = \sqrt{2} \sin \frac{n\pi}{2}, \quad n = 1, 3, 5, \dots$$

$$\omega_n^2 = (\beta_n \ell)^4 \frac{EI}{M \ell^3}, \quad \beta_n = \frac{n\pi}{\ell}, \quad n = 1, 3, 5, \dots$$

$$y(x, t) = \frac{2P_0 \ell^3}{EI} \sum_n \left\{ \frac{\sin \frac{n\pi}{2} \cdot \sin n\pi \frac{x}{\ell}}{(n\pi)^4} D_n(t) \right\} \quad n = 1, 3, 5, \dots$$

$$\underline{11-7} \quad \text{From Prob. 11-5} \quad K_m = \beta \ell \phi_m'(a), \quad \beta_m = \frac{n\pi}{\ell}$$

$$\phi_n(x) = \sqrt{2} \sin n\pi \frac{x}{\ell} \quad \frac{d\phi_n}{dx} = \sqrt{2} \frac{n\pi}{\ell} \cos \frac{n\pi x}{\ell}$$

$$\phi_n'(\frac{\ell}{2}) = \frac{1}{\beta_n} \frac{d\phi_n(\frac{\ell}{2})}{dx} = \frac{\sqrt{2}}{\ell} \cos \frac{n\pi}{2}, \quad n = 2, 4, 6, \dots$$

$$\ddot{q}_n + \omega_n^2 q_n = \frac{m_0}{M} [\beta_n \ell \phi_n'(a)] f(t)$$

$$q_n = \frac{m_0}{M \omega_n^2} \beta_n \ell \phi_n'(\frac{\ell}{2}) D_n(t)$$

$$y(x, t) = \frac{m_0}{M} \sum_n \frac{\beta_n \ell}{\omega_n^2} \phi_n(\frac{\ell}{2}) \phi_n(x) D_n(t) = \frac{m_0 \ell^3}{EI} \sum_n \frac{\phi_n'(\frac{\ell}{2}) \phi_n(x)}{(\beta_n \ell)^3} D_n(t)$$

$$= \frac{2m_0 \ell^2}{EI} \sum_{n=2,4,\dots} \frac{\cos \frac{n\pi}{2} \sin \frac{n\pi x}{\ell}}{(n\pi)^3} D_n(t)$$

11-8 Only even modes, $n = 2, 6, \dots$ will give $K_n \neq 0$

$$\phi_n = \sqrt{2} \sin \frac{n\pi x}{l}$$

$$K_n = \frac{1}{l} \int_0^{l/2} \sqrt{2} \sin \frac{n\pi x}{l} dx - \frac{1}{l} \int_{l/2}^l \sqrt{2} \sin \frac{n\pi x}{l} dx = \frac{2\sqrt{2}}{n\pi} (1 - \cos \frac{n\pi}{2})$$

$$D_n = \omega_n \int_0^t \sin \omega_n (t-\xi) d\xi = 1 - \cos \omega_n t$$

$$y(x, t) = \sum_n \frac{p_0 l K_n \phi_n}{\omega_n^2 M} D_n = \sum_{n=2, 6, \dots} \frac{p_0 l}{\omega_n^2 M} \frac{2\sqrt{2}}{n\pi} (1 - \cos \frac{n\pi}{2}) \sqrt{2} \sin \frac{n\pi x}{l} (1 - \cos \omega_n t)$$

$$= \frac{2p_0 l}{\pi M} \sum_{n=2, 6, \dots} 2 (1 - \cos \frac{n\pi}{2}) \sin \frac{n\pi x}{l} (1 - \cos \omega_n t)$$

1st mode, $n = 2$

$$y_1(x, t) = \frac{4p_0 l}{\pi M \omega_2^2} \sin \frac{2\pi x}{l} (1 - \cos \omega_2 t)$$

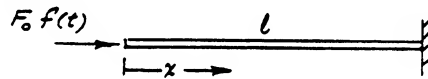
2nd mode, $n = 6$

$$y_2(x, t) = \frac{4p_0 l}{\pi M \omega_6^2} \sin \frac{6\pi x}{l} (1 - \cos \omega_6 t)$$

11-9

Normal mode $u(x) = A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}$

$$\frac{du}{dx} = \frac{\omega}{c} \{ A \cos \frac{\omega x}{c} - B \sin \frac{\omega x}{c} \}$$



Boundary Cond.

$$x = 0, \quad \frac{du}{dx} = 0 \quad \therefore A = 0$$

$$x = l, \quad u = 0 \quad \therefore B \cos \frac{\omega l}{c} = 0$$

$$\frac{\omega l}{c} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$u(x) = B \cos \frac{\omega x}{c}$$

$$\int_0^l \frac{M}{l} B^2 \cos^2 \frac{\omega x}{c} dx = M = \frac{B^2}{l} \int_0^l \frac{1}{2} (1 + \cos \frac{2\omega x}{c}) dx = \frac{B^2}{2}$$

$$\therefore B = \sqrt{2} \quad \phi_n(x) = \sqrt{2} \cos \frac{n\pi x}{2l}, \quad n = 1, 3, 5, \dots$$

11-9 Cont:

$$\ddot{q}_n + \omega_n^2 q_n = \frac{f(t)}{M} \int_0^l F_0 \delta(x) \sqrt{2} \cos \frac{n\pi x}{2l} dx = \frac{F_0 \sqrt{2}}{M} f(t)$$

$$q_n = \frac{F_0 K_n}{\omega_n^2 M} D_n(t) = \frac{\sqrt{2} F_0}{\left(\frac{n\pi}{2} \frac{c}{l}\right)^2 M} D_n(t) = \frac{\sqrt{2} F_0 M l^2}{\left(\frac{n\pi}{2}\right)^2 E A} D_n(t)$$

$$u(x, t) = \sum_n \phi_n(x) q_n(t) = \frac{2F_0 l}{EA} \left\{ \frac{\cos \frac{\pi x}{2l} \cdot D_1(t)}{\left(\frac{\pi}{2}\right)^2} + \frac{\cos \frac{3\pi x}{2l} \cdot D_3(t)}{\left(\frac{3\pi}{2}\right)^2} + \dots \right\}$$

11-10

$$K_n = \frac{1}{l} \int_0^l l \delta(x - \frac{l}{3}) \sqrt{2} \cos \frac{n\pi x}{2l} dx = \sqrt{2} \cos \frac{n\pi}{6}$$

$n = 1, 3, 5, \dots$

n	
1	$\cos \frac{\pi}{6} = \cos 30^\circ = .866$
3	$\cos \frac{3\pi}{6} = \cos 90^\circ = 0 \quad \therefore \text{mode absent}$
5	$\cos \frac{5\pi}{6} = \cos 150^\circ = -.866$
7	$\cos \frac{7\pi}{6} = \cos 210^\circ = -.866$
9	$\cos \frac{9\pi}{6} = \cos 270^\circ = 0 \quad \therefore \text{mode absent}$
11	$\cos \frac{11\pi}{6} = \cos 330^\circ = .866$

\therefore modes present are 1, 5, 7, 11, 13, ...

11-11

$$K_n = \sqrt{2} \cos \frac{n\pi}{6}$$

$$q_n = \frac{F_0 K_n}{\omega_n^2 M} D_n(t) \quad \omega_n^2 = \left(\frac{n\pi}{2} \frac{c}{l}\right)^2 = \left(\frac{n\pi}{2}\right)^2 \frac{EA}{Ml}$$

$$u(x, t) = \frac{2F_0 l}{AE} \left\{ \frac{0.866}{\left(\frac{\pi}{2}\right)^2} \cos \frac{\pi x}{2l} \cdot D_1(t) - \frac{0.866}{\left(\frac{5\pi}{2}\right)^2} \cos \frac{5\pi x}{2l} \cdot D_5(t) \right. \\ \left. - \frac{0.866}{\left(\frac{7\pi}{2}\right)^2} \cos \frac{7\pi x}{2l} \cdot D_7(t) + \frac{0.866}{\left(\frac{11\pi}{2}\right)^2} \cos \frac{11\pi x}{2l} \cdot D_{11}(t) + \dots \right\}$$

$$11-12 \quad y(x,t) = q_1 \sin \frac{\pi x}{\ell} + q_2$$

$$T = \frac{1}{2} \int_0^{\ell} m \dot{y}^2 dx = \frac{1}{2} \int_0^{\ell} (\dot{q}_1 \sin \frac{\pi x}{\ell} + \dot{q}_2)^2 dx$$

$$= \frac{1}{2} m \ell \left[\frac{1}{2} \dot{q}_1^2 + \frac{4}{\pi} \dot{q}_1 \dot{q}_2 + \dot{q}_2^2 \right]$$

$$U = \frac{1}{2} \left(\frac{k}{2} \right) y^2(0) + \frac{1}{2} \left(\frac{k}{2} \right) y^2(\ell) + \frac{1}{2} \int_0^{\ell} EI y''^2 dx$$

$$= \frac{1}{2} k q_2^2 + \frac{1}{2} EI q_1^2 \int_0^{\ell} \left(\frac{\pi}{\ell} \right)^4 \sin^2 \frac{\pi x}{\ell} dx$$

$$= \frac{1}{2} k q_2^2 + \frac{1}{2} EI q_1^2 \left(\frac{\pi}{\ell} \right)^4 \frac{\ell}{2} \quad \text{Sub. into } \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial U}{\partial q_i} = 0$$

$$\ddot{q}_1 + \frac{4}{\pi} \ddot{q}_2 + \pi \frac{EI}{m \ell^2} q_1 = 0, \quad \text{and} \quad \frac{2}{\pi} \ddot{q}_1 + \ddot{q}_2 + \frac{k}{m \ell} q_2 = 0$$

rewrite

$$\ddot{q}_1 + \frac{4}{\pi} \ddot{q}_2 + \omega_{11}^2 q_1 = 0 \quad \text{and} \quad \frac{2}{\pi} \ddot{q}_1 + \ddot{q}_2 + \omega_{22}^2 q_2 = 0$$

$$\left. \begin{aligned} (\omega_{11}^2 - \omega^2) q_1 - \frac{4}{\pi} \omega^2 q_2 &= 0 \\ -\frac{2}{\pi} \omega^2 q_1 + (\omega_{22}^2 - \omega^2) q_2 &= 0 \end{aligned} \right\} \quad \omega^4 - \left(\frac{\pi^2}{\pi^2 - 8} \right) (\omega_{11}^2 + \omega_{22}^2) \omega^2 + \frac{\pi \omega_{11}^2 \omega_{22}^2}{\pi^2 - 8} = 0$$

$$\omega^2 = \frac{\omega_{22}^2}{2} \left(\frac{\pi^2}{\pi^2 - 8} \right) \left[(1+R) \pm \sqrt{(1-R)^2 + \frac{32}{\pi^2} R} \right] \quad \text{where } R = \left(\frac{\omega_{11}}{\omega_{22}} \right)^2$$

Assume $y = (b + \sin \frac{\pi x}{\ell}) q$

$$T = \frac{1}{2} m \dot{q}^2 \int_0^{\ell} (b^2 + 2b \sin \frac{\pi x}{\ell} + \sin^2 \frac{\pi x}{\ell}) dx = \frac{1}{2} m \dot{q}^2 \left(b^2 \ell + 4b \frac{\ell}{\pi} + \frac{\ell}{2} \right)$$

$$U = \left[\frac{1}{2} k b^2 + \frac{1}{2} EI \left(\frac{\pi}{\ell} \right)^4 \frac{\ell}{2} \right] q^2$$

$$T = U \quad \text{gives}$$

$$\omega^2 = \frac{\frac{k}{m} b^2 + \frac{1}{2} \omega_{11}^2}{b^2 + \frac{4b}{\pi} + \frac{1}{2}}$$

Momentum = 0 gives

$$kb - m \omega^2 \int_0^{\ell} (\sin \frac{\pi x}{\ell} + b) dx = 0 \quad \therefore kb - m \omega^2 \left(-2 \frac{\ell}{\pi} + b \ell \right) = 0$$

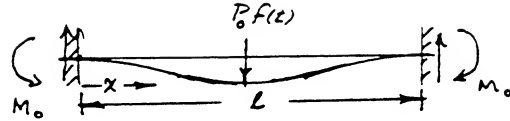
$$b = \frac{2}{\pi} \left(\frac{\omega^2}{\omega_{22}^2 - \omega^2} \right) = q_2 / q_1, \quad \text{or} \quad \omega^2 = \frac{\omega_{22}^2 b}{b + \frac{2}{\pi}}$$

$$\text{Equating } \omega^2 \quad \frac{\omega_{22}^2 b}{b + \frac{2}{\pi}} = \frac{\omega_{22}^2 b^2 + \frac{1}{2} \omega_{11}^2}{b^2 + \frac{4b}{\pi} + \frac{1}{2}} \quad \therefore b^2 + \frac{\pi}{4} (1-R) b + \frac{1}{2} R = 0$$

$$b = \frac{\pi}{8} \left[(R-1) \pm \sqrt{(R-1)^2 + \frac{32}{\pi^2} R} \right] \quad R = \left(\frac{\omega_{11}}{\omega_{22}} \right)^2$$

11-13

$$y = \sum_n \phi_n q_n$$



Generalized force

$$= P_0 f(t) \int_0^l \phi_n(x) \delta(x - \frac{l}{2}) dx = P_0 \phi_n(\frac{l}{2}) f(t)$$

$$y(\frac{l}{2}, t) = \sum_n \phi_n(\frac{l}{2}) q_n(t) \quad \text{where } q_n = \text{solution of}$$

$$\ddot{q}_n + \omega_n^2 q_n = \frac{P_0}{M} \phi_n(\frac{l}{2}) f(t)$$

Solution

$$q_n(t) = q_n(0) \cos \omega_n t + \frac{1}{\omega_n} \dot{q}_n(0) \sin \omega_n t + \frac{P_0 \phi_n(\frac{l}{2})}{M \omega_n^2} \omega_n \int_0^t f(\xi) \sin \omega_n (t - \xi) d\xi$$

From Appendix D

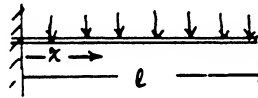
$$\phi_1(\frac{l}{2}) = 1.583, \quad \beta_1 l = 4.73 \quad \phi_1''(0) = 2.0$$

$$\phi_2(\frac{l}{2}) = 0 \quad \beta_2 l = 7.85 \quad \phi_2''(0) = 2.0$$

$$\phi_3(\frac{l}{2}) = -1.372 \quad \beta_3 l = 10.99 \quad \phi_3''(0) = 2.0$$

$$M_0 = EI \left(\frac{d^2 y}{dx^2} \right)_0 = EI \sum_n q_n(t) \left(\frac{d^2 \phi_n}{dx^2} \right)_{x=0} = EI \sum_n q_n(t) \beta_n^2 \phi_n''(0)$$

11-14



$$K_i = \frac{1}{l} \int_0^l \phi_i(x) dx$$

The above integral for K_i can be evaluated from tables in App. D. Add column of $\phi_i(x)$ and mult. by interval $\frac{dx}{l} = 0.04$. 1st mode column sum is 20.57 $\therefore 20.57 \times 0.04 = 0.822$. Exact value from integrating Eq. 9.4-12 with proper boundary conditions is 0.783. i.e. use $y(x) = (\cosh \beta x - \cos \beta x) - \left(\frac{\sinh \beta l - \sin \beta l}{\cosh \beta l + \cos \beta l} \right) (\sinh \beta x - \sin \beta x)$

Note: The number 20.57 is too large since it was computed with the largest deflection of $2.0 \times 0.04 + 1.89 \times 0.04 + \dots$ etc. If the sum is started with $1.89 \times 0.04 + \dots$ etc, then a figure of 18.57 is obtained. Averaging $\frac{1}{2}(18.57 + 20.57) \times 0.04$ we obtain the exact value 0.783.

Letting $\alpha_n = \frac{\sinh \beta l - \sin \beta l}{\cosh \beta l + \cos \beta l}$, the integral can be shown to

$$\text{be} \quad \int_0^l \phi_n(x) \frac{dx}{l} = \frac{2 \alpha_n}{\beta_n l}$$

11-15 From Eq. 11.2-8

$$\frac{1}{M} M(\omega_1^2 - \omega^2) = -k \phi_1^2(\frac{l}{3}) \frac{1}{\omega_1^2}$$

$$1 - (\frac{\omega}{\omega_1})^2 = -\frac{k}{M} \phi_1^2(\frac{l}{3}) \frac{1}{\omega_1^2} \quad (\frac{\omega}{\omega_1})^2 = 1 + \frac{k}{M} \phi_1^2(\frac{l}{3}) \frac{1}{\omega_1^2}$$

For simply supported beam $\phi_1(x) = \sqrt{2} \sin \frac{\pi x}{l}$

$$\therefore \phi_1^2(\frac{l}{3}) = 2 \sin^2 \frac{\pi}{3} = 1.50 \quad \omega_1^2 = \pi^4 \frac{EI}{Ml^3} \text{ for unconstrained beam.}$$

$$\therefore (\frac{\omega}{\omega_1})^2 = 1 + 1.5 \left(\frac{k}{M} \right) \left(\frac{Ml^3}{\pi^4 EI} \right)$$

11-16 From Eq. 11.2-8

$$\bar{q}_1 = \frac{1}{M(\omega_1^2 - \omega^2)} \left\{ -k \phi_1(\frac{l}{3}) \left[\bar{q}_1 \phi_1(\frac{l}{3}) + \bar{q}_2 \phi_2(\frac{l}{3}) \right] \right\}$$

$$\bar{q}_2 = \frac{1}{M(\omega_2^2 - \omega^2)} \left\{ -k \phi_2(\frac{l}{3}) \left[\bar{q}_1 \phi_1(\frac{l}{3}) + \bar{q}_2 \phi_2(\frac{l}{3}) \right] \right\}$$

Freq. Eq. becomes

$$\begin{vmatrix} \left[(\omega_1^2 - \omega^2) + \frac{k}{M} \phi_1^2(\frac{l}{3}) \right] & \frac{k}{M} \phi_1(\frac{l}{3}) \phi_2(\frac{l}{3}) \\ \frac{k}{M} \phi_1(\frac{l}{3}) \phi_2(\frac{l}{3}) & \left[(\omega_2^2 - \omega^2) + \frac{k}{M} \phi_2^2(\frac{l}{3}) \right] \end{vmatrix} = 0$$

11-17 We use here a somewhat different procedure than that of the text in that the influence coeffs. are found from the beam eq. $F(a, t) = -k y(\frac{l}{3})$

$$\alpha(a, x) = \alpha(\frac{l}{3}, \frac{l}{3}) = \frac{(\frac{l}{3})(\frac{2l}{3})}{6 EI l} (l^2 - \frac{4l^2}{9} - \frac{l^2}{9}) = \frac{4}{243} \frac{l^3}{EI}$$

From Eq. 11.4-4

$$y(\frac{l}{3}) = -k y(\frac{l}{3}) \frac{4}{243} \frac{l^3}{EI} + (\frac{\omega}{\omega_1})^2 q_1 \phi_1(\frac{l}{3}) + (\frac{\omega}{\omega_2})^2 q_2 \phi_2(\frac{l}{3}) + \dots$$

Using 1st mode only

$$y(\frac{l}{3}) \left[1 + \frac{4}{243} \frac{k l^3}{EI} \right] = (\frac{\omega}{\omega_1})^2 q_1 \phi_1(\frac{l}{3})$$

but $y(\frac{l}{3}) = \phi_1(\frac{l}{3}) q_1 + \text{higher modes which are neglected.}$

$$\therefore (\frac{\omega}{\omega_1})^2 = 1 + \frac{4}{243} \frac{k l^3}{EI}$$

Comparison with Prob. 11-15

$$\frac{1.5}{\pi^4} = 0.0154$$

$$\frac{4}{243} = 0.0165$$

11-18 From Eq. 11.3-4 with 1 mode

$$\phi_1(a) \bar{q}_1(t) = F(a, t) \alpha(a, a) - \left(\frac{\omega}{\omega_1}\right)^2 \bar{q}_1(t) \phi_1(a)$$

$$\phi_1(a) \bar{q}_1(t) = -k \phi_1(a) \bar{q}_1(t) \frac{\phi_1^2(a)}{M \omega_1^2} - \left(\frac{\omega}{\omega_1}\right)^2 \bar{q}_1(t) \phi_1(a)$$

$$\therefore \left(\frac{\omega}{\omega_1}\right)^2 = 1 + \frac{k}{M \omega_1^2} \phi_1^2(a)$$

11-19

$$\bar{q}_1(\omega_1^2 - \omega^2) = -\frac{k}{M} \varphi_1'(0) [\bar{q}_1 \varphi_1'(0) + \bar{q}_2 \varphi_2'(0)]$$

$$\bar{q}_2(\omega_2^2 - \omega^2) = -\frac{k}{M} \varphi_2'(0) [\bar{q}_1 \varphi_1'(0) + \bar{q}_2 \varphi_2'(0)]$$

Freq. eq. becomes

$$\begin{vmatrix} [(\omega_1^2 - \omega^2) + \frac{k}{M} \varphi_1'^2(0)] & \frac{k}{M} \varphi_1'(0) \varphi_2'(0) \\ \frac{k}{M} \varphi_1'(0) \varphi_2'(0) & [(\omega_2^2 - \omega^2) + \frac{k}{M} \varphi_2'^2(0)] \end{vmatrix} = 0$$

$\omega_2^2 = 16 \omega_1^2$ for simply supported beam. Let $\lambda = \left(\frac{\omega}{\omega_1}\right)^2$

$$[(1-\lambda) + \frac{k}{M \omega_1^2} \varphi_1'^2(0)] [(16-\lambda) + \frac{k}{M \omega_1^2} \varphi_2'^2(0)] - \left(\frac{k}{M \omega_1^2}\right)^2 [\varphi_1'(0) \varphi_2'(0)]^2 = 0$$

$$\lambda^2 - \left\{ 17 + \frac{k}{M \omega_1^2} [\varphi_1'^2(0) + \varphi_2'^2(0)] \right\} \lambda + \left\{ 16 + \frac{k}{M \omega_1^2} [\varphi_1'^2(0) + 16 \varphi_2'^2(0)] \right\} = 0$$

$$\varphi_1(x) = \sqrt{2} \sin \frac{\pi x}{\ell} \quad \varphi_1'(0) = \sqrt{2} \frac{\pi}{\ell} \quad \varphi_1'^2(0) = 2 \left(\frac{\pi}{\ell}\right)^2$$

$$\varphi_2(x) = \sqrt{2} \sin \frac{2\pi x}{\ell} \quad \varphi_2'(0) = 2\sqrt{2} \frac{\pi}{\ell} \quad \varphi_2'^2(0) = 8 \left(\frac{\pi}{\ell}\right)^2$$

If $k = 0$

$$\lambda^2 - 17\lambda + 16 = (\lambda - 1)(\lambda - 16) = 0$$

$$\therefore \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 16 \end{cases}$$

If $k \neq 0$ Let $\alpha = \frac{k}{M \omega_1^2}$

$$\lambda^2 - [17 + 10\left(\frac{\pi}{\ell}\right)^2 \alpha] \lambda + [16 + 40\left(\frac{\pi}{\ell}\right)^2 \alpha] = 0$$

11-21 Cont:

$$T = \int_0^l m \dot{\varphi}_0^2 \dot{q}_0^2 dm + \int_0^l 2m \varphi_0 \dot{q}_0 \dot{q}_1 dx + \int_0^l m \dot{\varphi}_1^2 \dot{q}_1^2 dx + \frac{1}{2} M_0 \dot{\varphi}_0^2 \dot{q}_0^2$$

$$= \frac{1}{2} \dot{q}_0^2 \{M_0 + 2ml\} + \{2ml \cdot .783\} \dot{q}_0 \dot{q}_1 + ml \dot{q}_1^2$$

$$U = \frac{1}{2} \int_0^l 2EI y''^2(x,t) dx = \frac{1}{2} \{2\omega^2 ml q_1^2\} \quad \text{see Eq. 11.1-7}$$

Lagrange's Eqs.

$$\ddot{q}_0 (M_0 + 2ml) + 2 \times .783 ml \ddot{q}_1 = 0$$

$$\ddot{q}_1 2ml + 2 \times .783 ml \ddot{q}_0 + 2\omega^2 ml q_1 = 0$$

Freq. Eq

$$\begin{vmatrix} -(M_0 + 2ml)\omega^2 & -2 \times .783 ml \omega^2 \\ -2 \times .783 ml \omega^2 & 2(\omega^2 - \omega_1^2) ml \end{vmatrix} = 0$$

Let $2ml = M$

$$-(M_0 + M)\omega^2(\omega^2 - \omega_1^2)M - (.783)^2 M^2 \omega^4 = 0$$

$$\frac{\omega}{\omega_1} = \sqrt{\frac{M_0 M + M^2}{M_0 M + .387 M^2}} \quad \text{where } \omega_1 = 1^{\text{st}} \text{ nat. freq. of cantilever beam of length } l \text{ and mass } \frac{1}{2} M$$

If $M_0 \rightarrow 0$, then $\omega = \omega_1 \frac{1}{\sqrt{.387}} = 1.61 \omega_1$

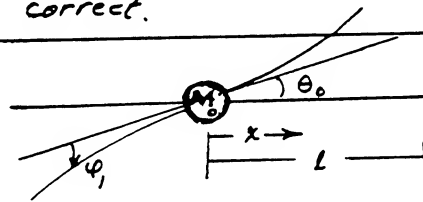
Since $\omega_1 = 3.52 \sqrt{\frac{EI}{ml^3}}$ and for free-free beam of length $2l$ has nat. freq $\omega_{ff} = 22.4 \sqrt{\frac{EI}{m(2l)^3}} = 5.57 \sqrt{\frac{EI}{ml^3}}$

$$\frac{5.57}{3.52} = 1.58, \quad \text{Since the airplane becomes a free-free beam of length } 2l \text{ as } M_0 \rightarrow 0, \text{ the above result is approximately correct.}$$

11-22

$$y(x,t) = q_0 \frac{x}{l} + \varphi_1 q_1$$

$$\theta_0 = \left(\frac{dy}{dx} \right)_{x=0} = \frac{q_0}{l}$$



$$T = \frac{1}{2} I_0 \left(\frac{\dot{q}_0}{l} \right)^2 + \frac{1}{2} \int_0^l 2m \left[\dot{q}_0 \frac{x}{l} + \varphi_1 \dot{q}_1 \right]^2 dx$$

$$= \frac{1}{2} I_0 \left(\frac{\dot{q}_0}{l} \right)^2 + \frac{1}{3} ml \dot{q}_0^2 + 2m \dot{q}_0 \dot{q}_1 \int_0^l \frac{x}{l} \varphi_1(x) dx + m \dot{q}_1^2 \int_0^l \varphi_1^2 dx$$

11-22 Cont:

$$U = \frac{1}{2} \{ 2\omega_1^2 (ml) \dot{q}_1^2 \} \quad \text{see Eq. 11.1-7}$$

Lagrange's Eq. & Characteristic eq.

$$\begin{vmatrix} -\left(\frac{I_0}{l^2} + \frac{2ml}{3}\right)\omega^2 & -\frac{2m\omega^2}{l} \int_0^l x \varphi_1 dx \\ -\frac{2m\omega^2}{l} \int_0^l x \varphi_1 dx & 2m(l\omega^2 - \omega^2 \int_0^l \varphi_1^2 dx) \end{vmatrix} = 0$$

For cantilever $\int_0^l \varphi_1^2 dx = l$, $\int_0^l x \varphi_1 dx = \frac{2}{\beta_1^2} = \frac{2l}{3.516} = \frac{l^2}{1.758}$

$$\omega^4 \left[2ml \left(\frac{I_0}{l^2} + \frac{2}{3} ml \right) - \frac{(2ml)^2}{3.08} \right] = \omega^2 \omega_1^2 2ml \left(\frac{I_0}{l^2} + \frac{2}{3} ml \right)$$

$$\omega = \omega_1 \sqrt{\frac{(I_0 + \frac{2}{3} ml^3)}{(I_0 + \frac{2}{3} ml^3) - \frac{2ml^3}{3.090}}}$$

To check for case $I_0 = 0$, the results should be the 2nd mode of a hinged-free beam of length l or 2nd mode of free-free beam of length $2l$.

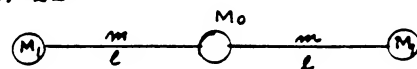
For $I_0 = 0$, $\omega = \omega_1 \sqrt{\frac{3.09}{.090}} = \omega_1 \sqrt{34.5} = 5.85 \omega_1$,

$$= 20.6 \sqrt{\frac{EI}{ml^3}} \quad * \text{ However correct result should be } 15.4 \sqrt{\frac{EI}{ml^3}}$$

* Difficulty due to difference of two numbers in denominator which is small. Needs 2nd mode φ_2 for better results.

11-23 See Prob. 11-21 and 11-22

T has added term $2(\frac{1}{2})M_1 \dot{y}^2(l, t)$



$$= M_1 [\varphi_0 \dot{q}_0 + \varphi_1(l) \dot{q}_1]^2 = M_1 \dot{q}_0^2 + 2M_1 \varphi_0 \varphi_1(l) \dot{q}_0 \dot{q}_1 + M_1 \varphi_1^2(l) \dot{q}_1^2$$

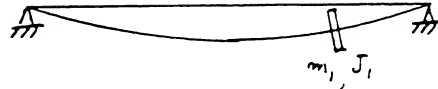
Lagrange's Eq. same as Prob 11-21 + $2M_1 [\ddot{q}_0 + \varphi_1(l) \ddot{q}_1] + 2M_1 [\varphi_1(l) \ddot{q}_0 + \varphi_1^2(l) \ddot{q}_1]$
New freq. eq.

$$\begin{vmatrix} -(M_0 + 2ml + 2M_1)\omega^2 & -(2x.783ml + 2M_1 \varphi_1(l))\omega^2 \\ -(2x.783ml + 2M_1 \varphi_1(l))\omega^2 & [2ml\omega^2 - \omega^2(2ml + 2M_1 \varphi_1^2(l))] \end{vmatrix} = 0$$

11-24

The additional mass changes

$$\text{only } T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} J_1 \dot{y}'^2 + \frac{1}{2} \int \dot{y}^2 dm$$



$$y(x, t) = \varphi_1(x) \dot{q}_1(t)$$

$$T = \frac{1}{2} m_1 \varphi_1^2(a) \dot{q}_1^2 + \frac{1}{2} J_1 \varphi_1'^2(a) \dot{q}_1^2 + \frac{1}{2} \int \varphi_1^2(x) dm \cdot \dot{q}_1^2$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} = m_1 \varphi_1^2(a) \ddot{q}_1 + J_1 \varphi_1'^2(a) \ddot{q}_1 + \ddot{q}_1 \int_0^l \varphi_1^2(x) dx$$

$$= [M_1 + m_1 \varphi_1^2(a) + J_1 \varphi_1'^2(a)] \ddot{q}_1$$

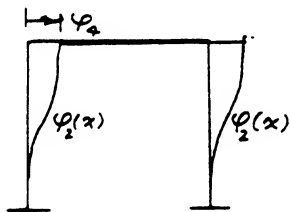
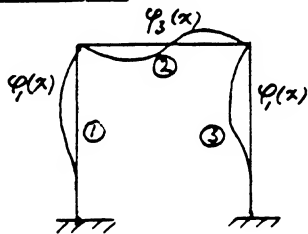
$$= M_1 \left[1 + \frac{m_1}{M_1} \varphi_1^2(a) + \frac{J_1}{M_1} \varphi_1'^2(a) \right] \ddot{q}_1 = M' \ddot{q}_1$$

$$M' \ddot{q}_1 + c_1 \dot{q}_1 + K_1 q_1 = \int_0^l p(x, t) \varphi_1(x) dx$$

$$\therefore \frac{c_1}{M_1} = 2\zeta' \omega_1 = \frac{c_1}{M_1 \left[1 + \frac{m_1}{M_1} \varphi_1^2(a) + \frac{J_1}{M_1} \varphi_1'^2(a) \right]}$$

$$\frac{K_1}{M_1} = \omega_1'^2 = \frac{K}{\text{same denominator}} \quad \text{etc.}$$

11-25



The modes are available in App. D.

$$\text{Member ① } w_1(x) = \varphi_1 p_1 + \varphi_2 p_2$$

$$\text{Member ② } w_2(x) = \varphi_3 p_3$$

$$u_2(x) = 1 p_4$$

$$\text{Member ③ } w_3(x) = \varphi_1 p_5 + \varphi_2 p_6$$

So there are 6 coordinates $p_1 \dots p_6$

Boundary conditions (4 eqs.)

$$w_1(l) = u_2(0) \quad \varphi_1(l) p_1 + \varphi_2(l) p_2 = p_4$$

$$w_1'(l) = w_2'(0)$$

$$\varphi_1'(l) p_1 + \varphi_2'(l) p_2 = \varphi_3'(0) p_3$$

$$w_2'(l) = w_3'(l)$$

$$\varphi_3'(l) p_3 = \varphi_1'(l) p_5 + \varphi_2'(l) p_6$$

$$u_2(l) = w_3(l)$$

$$p_4 = \varphi_1'(l) p_5 + \varphi_2'(l) p_6$$

11-25 Conti: Gen. mass & gen. stiffness from T & U

$$T = \frac{1}{2} \int_0^{l_1} \dot{w}_1^2 m dx + \frac{1}{2} \int_0^{l_2} [\dot{w}_2^2 + \dot{u}_2^2] m dx + \frac{1}{2} \int_0^{l_1} \dot{w}_3^2 m dx$$

$$U = \frac{1}{2} EI \int_0^{l_1} w_1''^2 dx + \frac{1}{2} EI \int_0^{l_2} w_2''^2 dx + \frac{1}{2} EI \int_0^{l_1} w_3''^2 dx$$

Mass Matrix

$$\begin{bmatrix} m_{11} & m_{12} & & & & \\ m_{21} & m_{22} & & & & \\ & & m_{33} & m_{34} & & \\ & & m_{43} & m_{44} & & \\ & & & & m_{55} & m_{56} \\ & & & & m_{65} & m_{66} \end{bmatrix}$$

Stiffness Matrix

$$\begin{bmatrix} k_{11} & k_{12} & & & & \\ k_{21} & k_{22} & & & & \\ & & k_{33} & k_{34} & & \\ & & k_{43} & k_{44} & & \\ & & & & k_{55} & k_{56} \\ & & & & k_{65} & k_{66} \end{bmatrix}$$

Constraint Eqs.

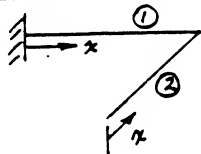
$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

2x6
Matrix

where q_1 & q_2 may be any of the p_s

System reduces to a 2×2 matrix equation.

11-26



Let $w_1(x) = \phi_1 p_1 + \phi_2 p_2 + \phi_3 p_3$

$\theta_1(x) = \phi_4 p_4$

$w_2(x) = \phi_5 p_5 + \phi_6 p_6 + \phi_7 p_7$

$\theta_2(x) = \phi_8 p_8 + \phi_9 p_9$

Subst. into

$$T = \frac{1}{2} \int \dot{w}^2 dm + \frac{1}{2} \int \frac{J}{A} \dot{\theta}^2 dm$$

$$U = \frac{1}{2} \int EI \left(\frac{d^2 w}{dx^2} \right)^2 dx + \frac{1}{2} \int C \left(\frac{d\theta}{dx} \right)^2 dx$$

to establish m_{ij} and k_{ij}

11-26 Cont:

Constraint eq. at junction of ① and ②

- (1) $w_1(l) = w_2(l)$ defl ① = defl ②
 (2) $\theta_1(l) = -w_2'(l)$ twist ① = bending slope ②
 (3) $C \theta_1'(l) = EI w_2''(l)$ Torque ① = bend. moment ②
 (4) $w_1'''(l) = -w_2'''(l)$ shear ① = shear ②
 (5) $EI w_1''(l) = -C \theta_2'(l)$ bend. mom. ① = torque at ②
 (6) $w_1'(l) = \theta_2(l)$ bend slope ① = twist ②

Mass Matrix

$$\begin{bmatrix}
 m_{11} & m_{12} & m_{13} & & & & & & \\
 m_{21} & m_{22} & m_{23} & & & & & & \\
 m_{31} & m_{32} & m_{33} & & & & & & \\
 & & & m_{44} & 0 & 0 & 0 & 0 & 0 \\
 & & & & m_{55} & m_{56} & m_{57} & 0 & 0 \\
 & & & & m_{65} & m_{66} & m_{67} & 0 & 0 \\
 & & & & m_{75} & m_{76} & m_{77} & 0 & 0 \\
 & & & & & & & m_{88} & m_{89} \\
 & & & & & & & m_{98} & m_{99}
 \end{bmatrix}
 \begin{matrix}
 \\ \\ \\
 \text{all 0} \\
 \\ \\ \\
 \end{matrix}
 \begin{matrix}
 \\ \\ \\
 \text{all 0} \\
 \\ \\ \\
 \end{matrix}
 = 9 \times 9$$

Stiffness Matrix

$$\begin{bmatrix}
 k_{11} & k_{12} & k_{13} & & & & & & \\
 k_{21} & k_{22} & k_{23} & & & & & & \\
 k_{31} & k_{32} & k_{33} & & & & & & \\
 & & & k_{44} & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & k_{77} & & \\
 & & & & & & & & k_{99}
 \end{bmatrix}
 \begin{matrix}
 \\ \\ \\
 \\ \\ \\
 \\ \\
 \end{matrix}
 = 9 \times 9$$

Constraint Matrix = 9×3

$$\begin{bmatrix}
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots
 \end{bmatrix}$$

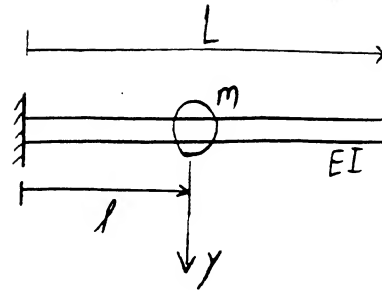
3 gen. coords. any 3 of the p_s

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Result is

$$\begin{bmatrix} M \\ 3 \times 3 \end{bmatrix}
 \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix}
 +
 \begin{bmatrix} K \\ 3 \times 3 \end{bmatrix}
 \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}
 = \begin{Bmatrix} 0 \end{Bmatrix}$$

11-27

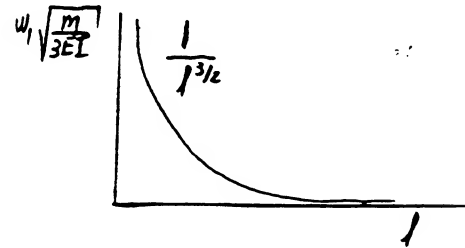


From the table of spring stiffness in chapter 2 at a distance l , $k = 3EI/l^3$

$$\therefore mg = ky \Rightarrow y = mg l^3 / 3EI$$

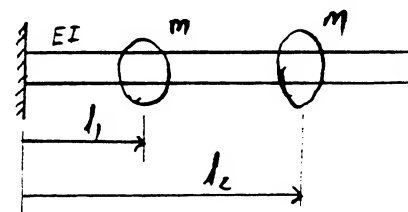
$$\Rightarrow \omega_1^2 = \frac{g y}{m y^2} = \frac{g}{y} = \frac{3EI}{m l^3} \quad \left(= \frac{k}{m} \right)$$

$$\therefore \omega_1 = \sqrt{3EI / m l^3}$$



11-28

The natural frequencies due to each mass are (see problem 11-27)

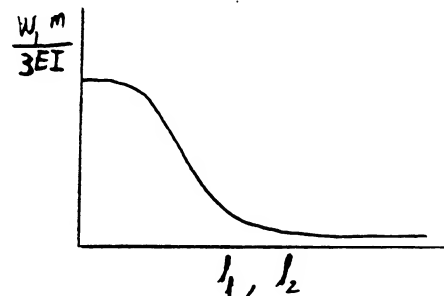


$$\omega_{n1}^2 = 3EI / m_1 l_1^3 \quad \text{and} \quad \omega_{n2}^2 = 3EI / m_2 l_2^3$$

Using Dunkerley's equation, we get

$$\omega_1^2 = \frac{\omega_{n1}^2 \omega_{n2}^2}{\omega_{n1}^2 + \omega_{n2}^2} = \left(\frac{3EI}{m} \right)^2 \frac{1/l_1^3 + 1/l_2^3}{1/l_1^3 + 1/l_2^3}$$

$$\therefore \omega_1 = \frac{3EI}{m} \sqrt{\frac{1}{l_1^3 + l_2^3}}$$



11-29

For problem 11-27 we have
 $k = \frac{3EI}{l^3}$

$$\therefore k - \omega^2 m = 0$$

$$\Rightarrow \omega^2 = \frac{k}{m} = \frac{3EI}{ml^3} \Rightarrow \omega = \sqrt{\frac{3EI}{ml^3}}$$

For problem 11-28 we have two nodes, one at l_1 and the other at l_2

The mass matrix is $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

Let the deflections of the two masses be y_1 and y_2
To find the stiffness matrix, we write

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \text{ where } F_1 \text{ and } F_2 \text{ are applied at the two masses}$$

$$\text{let } y_1 = 1 \text{ and } y_2 = 0 \Rightarrow F_1$$

12-1

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} (2m) \dot{x}_2^2$$

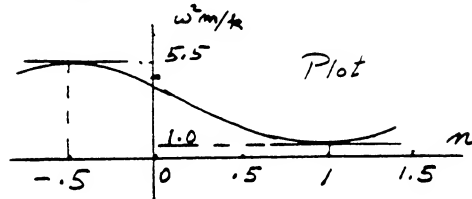
$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} (3k) (x_2 - x_1)^2 + \frac{1}{2} (2k) x_2^2$$

Let $\dot{x} = \omega x$ & equate

$$\omega^2 m x_1^2 + 2\omega^2 m x_2^2 = k x_1^2 + 3k (x_2 - x_1)^2 + 2k x_2^2$$

Let $x_2/x_1 = n$, then

$$\frac{\omega^2 m}{k} = \frac{1 + 3(n-1)^2 + 2n^2}{1 + 2n^2}$$

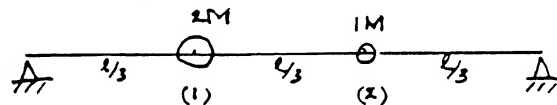


Can be checked by

$$\frac{\partial}{\partial n} \left(\frac{\omega^2 m}{k} \right) = 0 \rightarrow \text{gives } n = -0.5 \text{ and } n = 1.0$$

$$\text{Roots are } \frac{\omega^2 m}{k} = 1.0 \text{ and } \frac{\omega^2 m}{k} = 5.50$$

12-2



$$y(x) = \frac{Wbx}{6EI} (l^2 - x^2 - b^2) \quad 0 < x < (l-b)$$

Due to 2M

$$y_1 = \frac{(2Mg) \frac{2}{3}l \frac{1}{3}l}{6EI} \left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{2l}{3}\right)^2 \right] = \frac{16}{486} \left(\frac{Mgl^3}{EI} \right)$$

$$y_2 = \frac{(2Mg) \frac{1}{3}l \frac{1}{3}l}{6EI} \left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{l}{3}\right)^2 \right] = \frac{14}{486} \left(\frac{Mgl^3}{EI} \right)$$

Due to 1M

$$y_1 = \frac{Mg \frac{1}{3}l \frac{1}{3}l}{6EI} \left[l^2 - \left(\frac{l}{3}\right)^2 - \left(\frac{l}{3}\right)^2 \right] = \frac{7}{486} \left(\frac{Mgl^3}{EI} \right)$$

$$y_2 = \frac{Mg \frac{1}{3}l \frac{2}{3}l}{6EI} \left[l^2 - \left(\frac{2l}{3}\right)^2 - \left(\frac{l}{3}\right)^2 \right] = \frac{8}{486} \left(\frac{Mgl^3}{EI} \right)$$

$$\text{Add } y_1 = \frac{23}{486} \left(\frac{Mgl^3}{EI} \right) \quad y_2 = \frac{22}{486} \left(\frac{Mgl^3}{EI} \right)$$

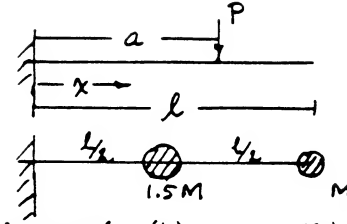
$$\omega_1^2 = \frac{g \left[(2M) \frac{23}{486} + (M) \frac{22}{486} \right]}{\left[2M \left(\frac{23}{486} \right)^2 + M \left(\frac{22}{486} \right)^2 \right]} \frac{1}{\frac{Mgl^3}{EI}} = 21.43 \frac{EI}{Ml^3}$$

$$\omega_1 = 4.63 \sqrt{\frac{EI}{Ml^3}} \text{ rad/s.}$$

12-3

$$y(x) = \frac{Px^2}{6EI} (3a-x) \quad 0 \leq x \leq a$$

$$= \frac{Pa^2}{6EI} (3x-a) \quad a \leq x \leq l$$



Due to $1.5M$ & M with results superimposed (1) (2)

$$y_1 = \frac{1.5Mg(\frac{l}{2})^2}{3EI} + \frac{Mg(\frac{l}{2})^2}{6EI} (3l - \frac{l}{2}) = \frac{16}{96} \frac{Mgl^3}{EI}$$

$$y_2 = \frac{1.5Mg(\frac{l}{2})^2}{6EI} (3l - \frac{l}{2}) + \frac{Mgl^3}{3EI} = \frac{47}{96} \frac{Mgl^3}{EI}$$

$$\omega^2 = \frac{[1.5(\frac{16}{96}) + (\frac{47}{96})] \cdot EI}{[1.5(\frac{16}{96})^2 + (\frac{47}{96})^2] Mgl^3} = 2.628 \frac{EI}{Ml^3}$$

$$\therefore \omega_1 = 1.621 \sqrt{\frac{EI}{Ml^3}}$$

12-4

$$a_{11} = \frac{1}{24} \frac{l^3}{EI}$$

$$a_{12} = a_{21} = \frac{10}{96} \frac{l^3}{EI}$$

$$a_{22} = \frac{l^3}{3EI}$$

$$\therefore [a] = \frac{l^3}{96EI} \begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix}$$

$$[k] = [a]^{-1} = \frac{96EI}{l^3} \frac{1}{28} \begin{bmatrix} 32 & -10 \\ -10 & 4 \end{bmatrix} = 13.7143 \frac{EI}{l^3} \begin{bmatrix} 8 & -2.5 \\ -2.5 & 1 \end{bmatrix}$$

Set up eq. of motion using flexibility

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{\omega^2 M l^3}{96EI} \begin{bmatrix} 4 & 10 \\ 10 & 32 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1.0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$= \frac{\omega^2 M l^3}{96EI} \begin{bmatrix} 6 & 10 \\ 15 & 32 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \quad \text{Let } \lambda = \frac{\omega^2 M l^3}{96EI}$$

$$\begin{bmatrix} (1-6\lambda) & -10\lambda \\ -15\lambda & (1-32\lambda) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

12-4 Cont.

$$42\lambda^2 - 38\lambda + 1 = 0$$

$$\lambda^2 - 0.9048\lambda + 0.0238 = 0$$

$$\lambda = 0.4524 \pm \sqrt{0.2048 - 0.0238} = 0.4524 \pm 0.4253$$

$$\lambda = \frac{\omega^2 M L^3}{96EI} = \begin{cases} 0.0271 \\ 0.8771 \end{cases} \quad \omega^2 = \begin{cases} 2.6016 \frac{EI}{ML^3} \\ 84.2576 \frac{EI}{ML^3} \end{cases}$$

$$\omega = \begin{cases} 1.6129 \sqrt{\frac{EI}{ML^3}} \\ 9.1792 \sqrt{\frac{EI}{ML^3}} \end{cases} \quad \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_1 = \frac{10\lambda}{1-6\lambda} = 0.3237$$

To verify $\omega^2 = \frac{X' K X}{X' M X}$

$$\omega^2 = \frac{(.3237 \ 1.0) \begin{bmatrix} 8 & -2.5 \\ -2.5 & 1 \end{bmatrix} \begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}}{(.3237 \ 1.0) \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}} = \frac{13.7143 \frac{EI}{L^3}}{2.605 \frac{EI}{ML^3}} = 2.605 \frac{EI}{ML^3}$$

$\therefore \omega_1 = 1.614 \sqrt{\frac{EI}{ML^3}}$

12-5

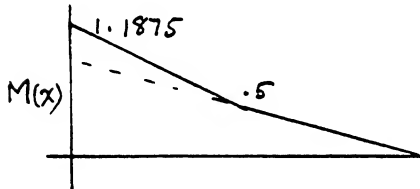
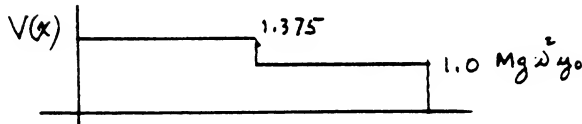
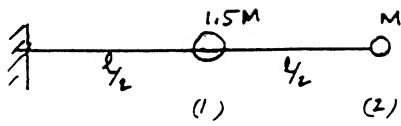
$$\omega^2 = \frac{X' M X}{X' M' a M X} =$$

$$= \frac{(.3237 \ 1.0) \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix} \cdot \frac{96EI}{ML^3}}{(.3237 \ 1.0) \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 10 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} .3237 \\ 1.0 \end{Bmatrix}}$$

$$= \frac{1.1572}{42.654} \cdot \frac{96EI}{ML^3} = 2.6045 \frac{EI}{ML^3}$$

$$\omega_1 = 1.6138 \sqrt{\frac{EI}{ML^3}}$$

12-6



$$\therefore V(x) = Mg \omega^2 y_0 \quad \frac{l}{2} \leq x \leq l$$

$$= 1.375 Mg \omega^2 y_0 \quad 0 \leq x \leq \frac{l}{2}$$

$$M(x) = \int_x^l V(\xi) d\xi = Mg \omega^2 y_0 \xi \Big|_x^l = Mg \omega^2 y_0 (l-x) \quad \frac{l}{2} \leq x \leq l$$

$$= Mg \omega^2 y_0 \left[\frac{l}{2} + 1.375 \left(\frac{l}{2} - x \right) \right] \quad 0 \leq x \leq \frac{l}{2}$$

$$U = \frac{1}{2} \int \frac{M^2}{EI} dx = \frac{(Mg \omega^2 y_0)^2}{2EI} \left[\int_{l/2}^l (l-x)^2 dx + \int_0^{l/2} \left\{ \frac{l}{2} + 1.375 \left(\frac{l}{2} - x \right) \right\}^2 dx \right]$$

$$= \frac{1}{2} \frac{(Mg \omega^2 y_0)^2}{EI} l^3 \left[\frac{1}{24} + \frac{7.0156}{24} \right] = \frac{1}{2} \frac{M^2 g^2 \omega^4 y_0^2}{EI} (0.4173)$$

$$T = \frac{1}{2} M \omega^2 y_0^2 + \frac{1}{2} (1.5M) \omega^2 y_0^2 \left(\frac{1}{2} \right)^2 = \frac{1}{2} M \omega^2 y_0^2 (1.0938)$$

$$\dot{y} = \omega y = \omega y_0 \left(\frac{x}{l} \right)^2$$

$$\ddot{y} = -\omega^2 y = -\omega^2 y_0 \left(\frac{x}{l} \right)^2$$

$$-m \ddot{y} = \omega^2 y \left(\frac{x}{l} \right)^2 m$$

For lumped mass, dynamic

Loads are

$$\omega^2 M y_0 g \quad \text{at } (2)$$

$$\text{and } \omega^2 1.5M y_0 g \left[\frac{1}{4} \right] +$$

$$\omega^2 M y_0 g [1]$$

$$\text{at } (1) = \omega^2 M g y_0 1.375$$

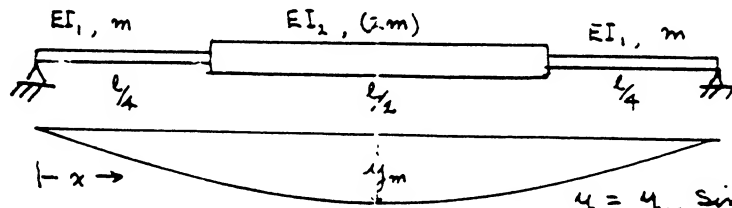
12-6 Cont

$$U = T \quad \text{gives}$$

$$\omega^2 = \frac{1.0938}{0.4173} \frac{EI}{ML^3} = 2.619 \frac{EI}{ML^3}$$

$$\omega_1 = 1.618 \sqrt{\frac{EI}{ML^3}} \quad \text{Exact val} = 1.6129 \sqrt{\frac{EI}{ML^3}}$$

12-7



$$y = y_m \sin \frac{\pi x}{l}$$

$$\frac{d^2 y}{dx^2} = -y_m \left(\frac{\pi}{l} \right)^2 \sin \frac{\pi x}{l}$$

$$U = \frac{1}{2} \int EI \left(\frac{d^2 y}{dx^2} \right)^2 dx = \frac{1}{2} y_m^2 \left(\frac{\pi}{l} \right)^4 \left[EI_1 \int_0^{l/4} \sin^2 \frac{\pi x}{l} dx + EI_2 \int_{l/4}^{l/2} \sin^2 \frac{\pi x}{l} dx \right]$$

$$= y_m^2 \left(\frac{\pi}{l} \right)^4 \left[EI_1 \frac{1}{2} \left(x - \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right) \Big|_0^{l/4} + EI_2 \frac{1}{2} \left(x - \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right) \Big|_{l/4}^{l/2} \right]$$

$$= y_m^2 \left(\frac{\pi}{l} \right)^4 l \left[.0454 EI_1 + .2046 EI_2 \right]$$

$$T = \frac{1}{2} \int m \dot{y}^2 dx = \frac{1}{2} \dot{\omega}^2 y_m^2 \left[m \int_0^{l/4} \sin^2 \frac{\pi x}{l} dx + (2m) \int_{l/4}^{l/2} \sin^2 \frac{\pi x}{l} dx \right]$$

$$= \omega^2 y_m^2 m l \left[.0454 + .4092 \right] = 0.4546 \omega^2 y_m^2 m l$$

Equating

$$\omega_1^2 = \frac{\pi^4}{m l^4} \left[0.100 EI_1 + 0.450 EI_2 \right]$$

$$\omega_1 = \pi^2 \sqrt{\frac{0.100 EI_1 + 0.450 EI_2}{m l^4}}$$

12-7 Cont:

(a) if $EI_2 = EI_1$ & $m_1 = m$, $m_2 = 2m$

$$\omega_1 = .7416 \pi^2 \sqrt{\frac{EI_1}{m l^4}} = 7.32 \sqrt{\frac{EI_1}{m l^4}}$$

(b) if $EI_2 = 4EI_1$,

$$\omega_1 = 1.3785 \pi^2 \sqrt{\frac{EI_1}{m l^4}}$$

12-8

$$y = y_0 \frac{x}{l} \left(1 - \frac{x}{l}\right) \quad \frac{dy}{dx} = \frac{y_0}{l} \left(1 - 2 \frac{x}{l}\right)$$

$$\frac{d^2 y}{dx^2} = -y_0 \frac{2}{l^2}$$

$$U = \frac{1}{2} \frac{4 y_0^2}{l^4} \left[2EI_1 \int_0^{l/4} dx + 2EI_2 \int_{l/4}^{l/2} dx \right] = \frac{1}{2} \left(\frac{4 y_0^2}{l^4} \right) \frac{l}{2} [EI_1 + EI_2]$$

$$= \frac{y_0^2}{l^3} [EI_1 + EI_2]$$

$$T = \frac{1}{2} \omega^2 y_0^2 \left[m \int_0^{l/4} \left(\frac{x}{l}\right)^2 \left(1 - \frac{x}{l}\right)^2 dx + (2m) \int_{l/4}^{l/2} \left(\frac{x}{l}\right)^2 \left(1 - \frac{x}{l}\right)^2 dx \right]$$

$$= \omega^2 y_0^2 \left[ml \left\{ \frac{1}{3} \left(\frac{x}{l}\right)^3 - \frac{1}{2} \left(\frac{x}{l}\right)^4 + \frac{1}{5} \left(\frac{x}{l}\right)^5 \right\}_0^{l/4} + 2ml \left\{ \text{same} \right\}_{l/4}^{l/2} \right]$$

$$= \omega^2 y_0^2 ml \left[(.008659) + (.0264) \right] = .03506 \omega^2 y_0^2 ml$$

Equating $T = U$

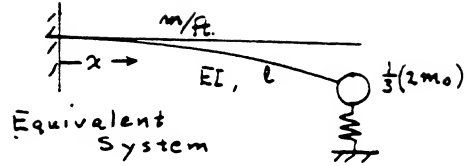
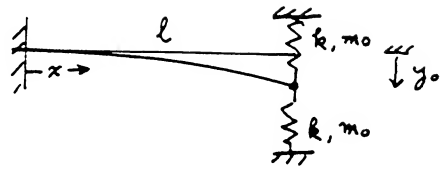
$$\omega_1^2 = 28.52 \left(\frac{EI_1 + EI_2}{m l^4} \right)$$

For $EI_2 = EI_1$, $\omega_1 = \sqrt{57.04} \sqrt{\frac{EI_1}{m l^4}} = 7.55 \sqrt{\frac{EI_1}{m l^4}}$

for Sine Curve Prob. 12-7

$$\omega_1 = 7.32 \sqrt{\frac{EI_1}{m l^4}}$$

12-9



$$\left. \begin{aligned} T_m &= \frac{1}{2} \left(\frac{33ml}{140} \right) \omega^2 y_0^2 + \frac{1}{2} \left(\frac{2}{3} m_0 \right) \omega^2 y_0^2 \\ U_m &= \frac{1}{2} \left(\frac{3EI}{l^3} \right) y_0^2 + \frac{1}{2} (2k) y_0^2 \end{aligned} \right\} \text{Equate}$$

$$\omega_1^2 = \frac{\frac{3EI}{l^3} + 2k}{\left(\frac{33ml}{140} + \frac{2}{3} m_0 \right)}$$

12-10

$$y_m = \sin \frac{\pi x}{l} + b$$

$$U = \frac{1}{2} EI \int_0^l (y'')^2 dx + \frac{1}{2} \left(\frac{k}{2} \right) y_{x=0}^2 + \frac{1}{2} \left(\frac{k}{2} \right) y_{x=l}^2$$

$$U_{max} = \frac{1}{2} EI \int_0^l \left(\frac{\pi}{l} \right)^2 \sin^2 \frac{\pi x}{l} dx + \frac{1}{2} k b^2$$

$$\begin{aligned} T_{max} &= \frac{1}{2} \int_0^l \frac{M}{l} \dot{y}^2 dx = \frac{1}{2} \omega^2 \frac{M}{l} \int_0^l \left(\sin^2 \frac{\pi x}{l} + 2b \sin \frac{\pi x}{l} + b^2 \right) dx \\ &= \frac{M\omega^2}{2} \left(\frac{1}{2} + \frac{4b}{\pi} + b^2 \right) \end{aligned}$$

$$\text{Equating } U_{max} = T_{max} \quad \omega_1^2 = \frac{2k}{M} \left(\frac{\frac{K}{2} \frac{\pi^4}{4} + \frac{b^2}{2}}{\frac{1}{2} + \frac{4b}{\pi} + b^2} \right)$$

$$\text{where } K = EI/l^3$$

$$\frac{\partial \omega_1^2}{\partial b} = 0 \quad \text{gives}$$

$$b = -\frac{\pi}{4} \left(\frac{1}{2} - \frac{K\pi^4}{2k} \right) \pm \sqrt{\left[\frac{\pi}{4} \left(\frac{1}{2} - \frac{K\pi^4}{2k} \right) \right]^2 + \frac{\pi^4 K}{2k}}$$

12-11

$$y = y_m \left[3 \left(\frac{x}{l} \right) - 4 \left(\frac{x}{l} \right)^3 \right] \quad 0 \leq x \leq \frac{l}{2}$$

$$m(x) = m_0 \left(\frac{x}{l} \right) \left(1 - \frac{x}{l} \right)$$

$$T_m = \frac{1}{2} \times 2 \int_0^{l/2} m(x) \omega^2 y^2 dx = m_0 \omega^2 y_m^2 \int_0^{l/2} \frac{x}{l} \left[9 \left(\frac{x}{l} \right)^2 - 24 \left(\frac{x}{l} \right)^4 + 16 \left(\frac{x}{l} \right)^6 \right] dx$$

$$= m_0 \omega^2 y_m^2 l \left[\frac{9}{4} \left(\frac{1}{2} \right)^4 - \frac{24}{6} \left(\frac{1}{2} \right)^6 + \frac{16}{8} \left(\frac{1}{2} \right)^8 - \frac{9}{5} \left(\frac{1}{2} \right)^5 + \frac{24}{7} \left(\frac{1}{2} \right)^7 - \frac{16}{9} \left(\frac{1}{2} \right)^9 \right]$$

$$= 0.0529 m_0 l \omega^2 y_m^2$$

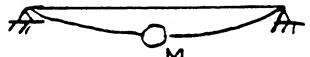
$$U_m = \frac{1}{2} \times 2 \int_0^{l/2} EI \left(\frac{d^2 y}{dx^2} \right)^2 dx = y_m^2 EI \int_0^{l/2} \frac{24}{l^6} x^2 dx = y_m^2 EI \frac{24}{l^6} \frac{1}{3} \left(\frac{l}{2} \right)^3$$

Equating

$$\omega_1^2 = 453 \frac{EI}{(m_0 l) l^3} \quad \omega_1 = 21.30 \sqrt{\frac{EI}{(m_0 l) l^3}}$$

To check, first find total mass

$$\int m dx = 2 m_0 \int_0^{l/2} \left(\frac{x}{l} - \frac{x^3}{l^3} \right) dx = 2 m_0 \left[\frac{x^2}{2l} - \frac{x^4}{4l^3} \right]_0^{l/2} = 0.2188 m_0 l = M_T$$

 For massless beam with M at midspan $k = \frac{48EI}{l^3}$

$$T = \frac{1}{2} m_{\text{eff}} \dot{\phi}^2 y_m^2 = 0.0529 m_0 l \dot{\phi}^2 y_m^2 \text{ from above}$$

$$\therefore m_{\text{eff}} = 0.1058 m_0 l = N M_T = N \cdot 0.2188 m_0 l$$

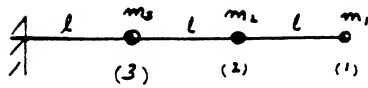
$$\therefore N = 0.4835 \quad \therefore m_{\text{eff}} = 0.4835 (\text{total mass})$$

$$\omega^2 = \frac{k}{m_{\text{eff}}} = \frac{48EI}{0.4835 M_T l^3} = 99.28 \frac{EI}{M_T l^3}$$

$$\omega = 9.96 \sqrt{\frac{EI}{M_T l^3}} \quad \therefore \text{results appear to be reasonable}$$

12-12

From Eq. 12.2-3, $\frac{1}{\omega^2} \cong a_{11} m_1 + a_{22} m_2 + a_{33} m_3$



$$a_{11} = \frac{(3l)^3}{3EI} = \frac{27l^3}{3EI}$$

$$a_{22} = \frac{(2l)^3}{3EI} = \frac{8l^3}{3EI}$$

$$a_{33} = \frac{l^3}{3EI}$$

$$\therefore \frac{1}{\omega^2} \cong (27m_1 + 8m_2 + m_3) \frac{l^3}{3EI} \quad \omega_1 \cong \sqrt{\frac{3EI}{l^3} \left(\frac{1}{27m_1 + 8m_2 + m_3} \right)}$$

12-13

Flex. $a_{xb} = \frac{bx}{6EI} (l^2 - x^2 - b^2)$

$$a_{11} = \frac{\frac{3}{4}l \cdot \frac{1}{4}l}{6EI} \left(l^2 - \frac{l^2}{16} - \frac{9}{16}l^2 \right) = \frac{3}{16^2} \frac{l^3}{EI} = a_{33}$$

$$a_{22} = \frac{\frac{1}{2}l \cdot \frac{1}{2}l}{6EI} \left(l^2 - \frac{l^2}{4} - \frac{l^2}{4} \right) = \frac{1}{48} \frac{l^3}{EI} = \frac{16}{3} \cdot \frac{l^3}{16^2 EI}$$

$$\frac{1}{\omega^2} \cong a_{11} \frac{W_1}{g} + a_{22} \frac{W_2}{g} + a_{33} \frac{W_3}{g}$$

$$= \frac{l^3}{16^2 EI g} \left(3W_1 + \frac{16}{3}W_2 + 3W_3 \right) = \frac{l^3 W}{16^2 EI} \left(3 + \frac{16}{3} \times 4 + 6 \right)$$

$$\therefore \omega_1 \cong 16 \sqrt{\frac{3}{91} \frac{EI g}{W l^3}} = 2.905 \sqrt{\frac{EI g}{W l^3}}$$

12-14

Use Eq. (b) Ex 12.2-1 with $a_{22} = \frac{0.78}{1000}$

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{11}^2} + \frac{0.78}{1000} \frac{320}{386} = \left(\frac{60}{2\pi \times 622} \right)^2 + 0.6466 \times 10^{-3}$$

$$= (0.2357 + 0.6466) \times 10^{-3}$$

$$= 0.8823 \times 10^{-3}$$

$$\omega_1 \cong \sqrt{\frac{1}{0.8823 \times 10^{-3}}} = 33.66 \text{ rad/sec} = 5.358 \text{ Hz}$$

12-15

$$\frac{1}{(\omega_1)_1^2} - \frac{1}{\omega_{11}^2} = a_{22}(m_2)_1$$

$$\frac{1}{(\omega_1)_2^2} - \frac{1}{\omega_{11}^2} = a_{22}(m_2)_2 \quad \text{divide to eliminate } a_{22}$$

$$\frac{\omega_{11}^2 - (\omega_1)_1^2}{(\omega_1)_1^2} \cdot \frac{(\omega_1)_2^2}{\omega_{11}^2 - (\omega_1)_2^2} = \frac{(m_2)_1}{(m_2)_2}$$

$$(\omega_1)_1 = 435 \times 2\pi \quad (m_2)_1 = 5.44$$

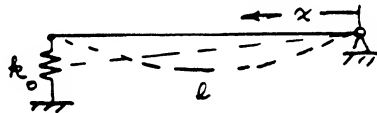
$$(\omega_1)_2 = 398 \times 2\pi \quad (m_2)_2 = 5.44 + 4.52 = 9.96$$

$$\left(\frac{398}{435}\right)^2 \left[\frac{f_{11}^2 - 435^2}{f_{11}^2 - 398^2} \right] = \frac{5.44}{9.96} \quad \therefore f_{11}^2 = 245219$$

$$f_{11} = 495.2 \text{ cps}$$

12-16

$$\phi_1 = \frac{x}{l}, \quad \phi_2 = \sin \frac{\pi x}{l}$$



$$m_{11} = \int_0^l m \phi_1 \phi_1 dx = \frac{m}{l^2} \int_0^l x^2 dx = \frac{ml}{3}$$

$$m_{12} = \int_0^l m \frac{x}{l} \cdot \sin \frac{\pi x}{l} dx = \frac{ml}{\pi^2} \int_0^{\frac{\pi x}{l} = \pi} \left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi x}{l}\right) d\left(\frac{\pi x}{l}\right)$$

$$= \frac{ml}{\pi^2} \left[\sin \frac{\pi x}{l} - \left(\frac{\pi x}{l}\right) \cos \frac{\pi x}{l} \right]_0^{\frac{\pi x}{l} = \pi} = \frac{ml}{\pi}$$

$$m_{22} = m \int_0^l \sin^2 \frac{\pi x}{l} dx = m \int_0^l \frac{1}{2} \left[1 - \cos \frac{2\pi x}{l} \right] dx = \frac{ml}{2}$$

12-16 Cont.

$$EI \int_0^l \phi_1'' \phi_1'' dx = 0 \quad EI \int_0^l \phi_1'' \phi_2'' dx = 0$$

$$EI \int_0^l \phi_2'' \phi_2'' dx = EI \left(\frac{\pi}{l}\right)^4 \int_0^l \sin^2 \frac{\pi x}{l} dx = \left(\frac{\pi}{l}\right)^4 EI \frac{l}{2}$$

$$u = C_1 \phi_1(x) + C_2 \phi_2(x)$$

$$U = \frac{1}{2} \int_0^l EI (u'')^2 dx + \frac{1}{2} k_0 u^2(l) \quad \text{where } u^2(l) = C_1^2$$

$$= \frac{1}{2} k_0 C_1^2 + \frac{1}{2} \left(\frac{\pi}{l}\right)^4 EI \frac{l}{2} C_2^2$$

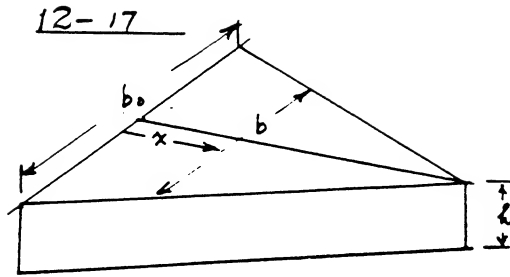
$$\frac{\partial U}{\partial C_1} = k_0 C_1$$

$$\frac{\partial U}{\partial C_2} = \left(\frac{\pi}{l}\right)^4 EI \frac{l}{2} C_2$$

∴ Eq. 12-7 becomes

$$\begin{bmatrix} (k_0 - \omega^2 \frac{m l}{3}) & -\omega^2 \frac{m l}{\pi} \\ -\omega^2 \frac{m l}{\pi} & \left\{ \left(\frac{\pi}{l}\right)^4 EI \frac{l}{2} - \omega^2 \frac{m l}{2} \right\} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[\left(\frac{1}{6} - \frac{1}{\pi^2} \right) (m l)^2 \right] \omega^4 - m l \left[\frac{\pi^4 EI}{6 l^3} + \frac{k_0}{2} \right] \omega^2 + \frac{k_0 \pi^4 EI}{2 l^3} = 0$$



$$m(x) = m_0 \left(1 - \frac{x}{l}\right)$$

$$b(x) = b_0 \left(1 - \frac{x}{l}\right)$$

$$I(x) = \frac{b l^3}{12} = \frac{l^3}{12} b_0 \left(1 - \frac{x}{l}\right)$$

$$\therefore EI = EI_0 \left(1 - \frac{x}{l}\right)$$

$$y = C_1 x^2 + C_2 x^3 = C_1 \phi_1 + C_2 \phi_2$$

$$\phi_1 = x^2$$

$$\phi_2 = x^3$$

$$k_{ij} = \int_0^l EI \phi_i'' \phi_j'' dx$$

$$\phi_1'' = 2$$

$$\phi_2'' = 6x$$

$$m_{ij} = \int m(x) \phi_i \phi_j dx$$

$$k_{11} = 4EI_0 \int_0^l \left(1 - \frac{x}{l}\right) dx = 4EI_0 \left(l - \frac{l}{2}\right) = 2EI_0 l$$

$$k_{12} = 12EI_0 \int_0^l x \left(1 - \frac{x}{l}\right) dx = 12EI_0 \left(\frac{l^2}{2} - \frac{l^3}{3l}\right) = 2EI_0 l^2$$

$$k_{22} = 36EI_0 \int_0^l x^2 \left(1 - \frac{x}{l}\right) dx = 36 \left(\frac{l^3}{3} - \frac{l^4}{4l}\right) = 3EI_0 l^3$$

$$m_{11} = m_0 \int_0^l x^4 \left(1 - \frac{x}{l}\right) dx = m_0 \left(\frac{l^5}{5} - \frac{l^6}{6l}\right) = \frac{1}{30} m_0 l^5$$

$$m_{12} = m_0 \int_0^l x^5 \left(1 - \frac{x}{l}\right) dx = m_0 \left(\frac{l^6}{6} - \frac{l^7}{7l}\right) = \frac{1}{42} m_0 l^6$$

$$m_{22} = m_0 \int_0^l x^6 \left(1 - \frac{x}{l}\right) dx = m_0 \left(\frac{l^7}{7} - \frac{l^8}{8l}\right) = \frac{1}{56} m_0 l^7$$

12-17 Cont.

Subst. into Eq. 12.3-7

$$\begin{bmatrix} (2EI_0 l - \omega^2 \frac{m l^5}{30}) & (2EI_0 l^2 - \omega^2 \frac{m_0 l^6}{42}) \\ (2EI_0 l^2 - \omega^2 \frac{m_0 l^6}{42}) & (3EI_0 l^3 - \omega^2 \frac{m_0 l^7}{56}) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Freq. Eq. from determinant of above eq = 0

$$6(EI_0)^2 l^4 - \omega^2 \left[\frac{m_0 l^5}{30} 3EI_0 l^3 + \frac{m_0 l^7}{56} 2EI_0 l \right] + \omega^4 \left(\frac{m_0 l^5}{30} \cdot \frac{m_0 l^7}{56} \right) - 4(EI_0)^2 l^4 + \left[4EI_0 l^2 \frac{m_0 l^6}{42} \right] \omega^2 - \omega^4 \left(\frac{m_0^2 l^{12}}{42^2} \right) = 0$$

$$\omega^4 \left[m_0^2 l^{12} \left(\frac{1}{30 \times 56} - \frac{1}{42^2} \right) \right] - \omega^2 \left[m_0 EI_0 l^8 \left(\frac{3}{30} + \frac{2}{56} - \frac{4}{42} \right) \right] + 2(EI_0)^2 l^4 = 0$$

$$(28.345 \times 10^{-6} m_0^2 l^{12}) \omega^4 - (0.0405 m_0 EI_0 l^3) \omega^2 + 2(EI_0)^2 l^4 = 0$$

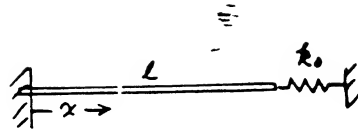
$$\omega^4 - 1429.3 \left(\frac{EI_0}{m_0 l^4} \right) \omega^2 + 70559 \left(\frac{EI_0}{m_0 l^4} \right)^2 = 0$$

$$\omega^2 = \left\{ \begin{array}{c} 51.20 \\ 1378 \end{array} \right\} \times \frac{EI_0}{m_0 l^4}$$

$$\omega_1 = 7.155 \sqrt{\frac{EI_0}{m_0 l^4}}$$

$$\omega_2 = 37.12 \sqrt{\frac{EI_0}{m_0 l^4}}$$

12-18



Normal modes of fixed-free uniform rods are

$$\phi_1 = \sin \frac{\pi}{2} \frac{x}{L}$$

$$\phi_2 = \sin \frac{3\pi}{2} \frac{x}{L} \quad \text{See Eq. 9.2-8}$$

$$\therefore u(x) = C_1 \sin \frac{\pi}{2} \frac{x}{L} + C_2 \sin \frac{3\pi}{2} \frac{x}{L}$$

$$\frac{\partial u}{\partial x} = C_1 \frac{\pi}{2L} \cos \frac{\pi}{2} \frac{x}{L} + C_2 \frac{3\pi}{2L} \cos \frac{3\pi}{2} \frac{x}{L}$$

$$U = \frac{1}{2} AE \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx + \frac{1}{2} k_0 u^2(L)$$

$$= \frac{1}{2} AE \int_0^L C_1^2 \left(\frac{\pi}{2L} \right)^2 \cos^2 \frac{\pi}{2} \frac{x}{L} dx + AE C_1 C_2 \int_0^L \left(\frac{\pi}{2L} \right) \left(\frac{3\pi}{2L} \right) \cos \frac{\pi}{2} \frac{x}{L} \cos \frac{3\pi}{2} \frac{x}{L} dx$$

$$+ \frac{1}{2} AE \int_0^L C_2^2 \left(\frac{3\pi}{2L} \right)^2 \cos^2 \frac{3\pi}{2} \frac{x}{L} dx + \frac{1}{2} k_0 (C_1 - C_2)^2$$

$$\frac{\partial U}{\partial C_1} = \left[AE \left(\frac{\pi}{2L} \right)^2 \int_0^L \cos^2 \frac{\pi}{2} \frac{x}{L} dx + k_0 \right] C_1 - k_0 C_2$$

$$+ AE C_2 \int_0^L \left(\frac{\pi}{2L} \right) \left(\frac{3\pi}{2L} \right) \cos \frac{\pi}{2} \frac{x}{L} \cos \frac{3\pi}{2} \frac{x}{L} dx \quad \rightarrow 0$$

$$\frac{\partial U}{\partial C_2} = \left[AE \left(\frac{3\pi}{2L} \right)^2 \int_0^L \cos^2 \frac{3\pi}{2} \frac{x}{L} dx \right] C_2 - k_0 C_1 + k_0 C_2$$

$$\therefore k_{11} = AE \left(\frac{\pi}{2L} \right)^2 \int_0^L \cos^2 \frac{\pi}{2} \frac{x}{L} dx + k_0$$

$$k_{12} = k_{21} = -k_0$$

$$k_{22} = AE \left(\frac{3\pi}{2L} \right)^2 \int_0^L \cos^2 \frac{3\pi}{2} \frac{x}{L} dx + k_0$$

12-18 Cont.

$$\text{Since } \int_0^l \cos^2 n\theta d\theta = \frac{l}{2} \quad n = \frac{\pi}{2l}, \frac{3\pi}{2l},$$

$$k_{11} = AE\left(\frac{\pi}{2l}\right)^2 \frac{l}{2} + k_0 \quad k_{12} = -k_0$$

$$k_{22} = AE\left(\frac{3\pi}{2l}\right)^2 \frac{l}{2} + k_0$$

$$m_{ij} = \int_0^l m \phi_i \phi_j dx = \begin{cases} m_{11} = \frac{ml}{2} \\ m_{12} = 0 \\ m_{22} = \frac{ml}{2} \end{cases}$$

Subst into Eq. 12.3-7

$$\begin{bmatrix} \left\{ AE\left(\frac{\pi}{2l}\right)^2 \frac{l}{2} + k_0 - \omega^2 \frac{ml}{2} \right\} & -k_0 \\ -k_0 & \left\{ AE\left(\frac{3\pi}{2l}\right)^2 \frac{l}{2} + k_0 - \omega^2 \frac{ml}{2} \right\} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Freq. Eq.

$$\begin{aligned} \left(\frac{ml}{2} \omega^2\right)^2 - \left(\frac{ml}{2} \omega^2\right) \left[AE\left(\frac{\pi}{2l}\right)^2 \frac{l}{2} + k_0 + AE\left(\frac{3\pi}{2l}\right)^2 \frac{l}{2} + k_0 \right] \\ + \left[AE\left(\frac{\pi}{2l}\right)^2 \frac{l}{2} + k_0 \right] \left[AE\left(\frac{3\pi}{2l}\right)^2 \frac{l}{2} + k_0 \right] - k_0^2 = 0 \end{aligned}$$

Reduces to

$$\omega^4 - \omega^2 \left[10 \left(\frac{\pi}{2l}\right)^2 \frac{AE}{m} + \frac{4k_0^2}{ml} \right] + \left[9 \left(\frac{\pi}{2l}\right)^4 \left(\frac{AE}{m}\right)^2 + 20 \left(\frac{\pi}{2l}\right)^2 \frac{AE}{m} \frac{k_0}{ml} \right] = 0$$

12-19 Refer to Prob 12-18

With $k_0 = 0$

$$k_{11} = AE \left(\frac{\pi}{2l} \right)^2 \frac{l}{2} \quad k_{12} = 0$$

$$k_{22} = AE \left(\frac{3\pi}{2l} \right)^2 \frac{l}{2}$$

$$\text{Additional KE} = \frac{1}{2} m_0 \dot{u}(l)^2 = \frac{1}{2} m_0 (\dot{c}_1 - \dot{c}_2)^2$$

$$= \frac{1}{2} m_0 (\dot{c}_1^2 - 2\dot{c}_1\dot{c}_2 + \dot{c}_2^2)$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{c}_1} = m_0 \ddot{c}_1 - m_0 \ddot{c}_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{c}_2} \right) = -m_0 \ddot{c}_1 + m_0 \ddot{c}_2$$

$$\therefore m_{11} = \frac{m l}{2} + m_0$$

$$m_{12} = m_{21} = -m_0$$

$$m_{22} = \frac{m l}{2} + m_0$$

Eq. 12.3-7 becomes

$$\begin{bmatrix} \left\{ AE \left(\frac{\pi}{2l} \right)^2 \frac{l}{2} - \omega^2 \left(\frac{m l}{2} + m_0 \right) \right\} & \omega^2 m_0 \\ \omega^2 m_0 & \left\{ AE \left(\frac{3\pi}{2l} \right)^2 \frac{l}{2} - \omega^2 \left(\frac{m l}{2} + m_0 \right) \right\} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

12-20

$$m(x) = m_0 \frac{x}{l} \left(1 - \frac{x}{l}\right)$$

$$\phi_1 = \sin \frac{\pi x}{l}$$

$$\phi_2 = \sin \frac{2\pi x}{l}$$

$$y = \phi_1 q_1 + \phi_2 q_2$$

$$k_{11} = EI \int_0^l \phi_1'' \phi_1'' dx = EI \left(\frac{\pi}{l}\right)^4 \int_0^l \sin^2 \frac{\pi x}{l} dx = EI \left(\frac{\pi}{l}\right)^4 \frac{l}{2}$$

$$k_{12} = k_{21} = EI \int_0^l \phi_1'' \phi_2'' dx = 0$$

$$k_{22} = EI \int_0^l \phi_2'' \phi_2'' dx = EI \left(\frac{2\pi}{l}\right)^4 \int_0^l \sin^2 \frac{2\pi x}{l} dx = EI \left(\frac{2\pi}{l}\right)^4 \frac{l}{2}$$

$$m_{11} = \int_0^l m(x) \phi_1 \phi_1 dx = m_0 \int_0^l \frac{x}{l} \left(1 - \frac{x}{l}\right) \sin^2 \frac{\pi x}{l} dx$$

$$= m_0 \int_0^l \frac{x}{l} \frac{1}{2} \left(1 - \cos \frac{2\pi x}{l}\right) dx - m_0 \int_0^l \left(\frac{x}{l}\right)^2 \frac{1}{2} \left(1 - \cos \frac{2\pi x}{l}\right) dx$$

$$= m_0 \left[\frac{l}{4} - \frac{l}{2\pi^2} \left(\frac{\cos(2\pi x/l)}{(2\pi/l)^2} - \left(\frac{\pi x}{l}\right) \frac{\sin(2\pi x/l)}{(2\pi/l)} \right) \right]_0^l$$

$$- m_0 \left[\frac{l}{6} - \frac{l}{16\pi^3} \int_0^l \left(\frac{2\pi x}{l}\right)^2 \cos \frac{2\pi x}{l} d\left(\frac{2\pi x}{l}\right) \right]$$

$$= \frac{m_0 l}{12} + \frac{m_0 l}{16\pi^3} \left[2 \left(\frac{2\pi x}{l}\right) \cos \frac{2\pi x}{l} + \left\{ \left(\frac{2\pi x}{l}\right)^2 - 2 \right\} \sin \frac{2\pi x}{l} \right]_0^l$$

$$= \frac{m_0 l}{12} + \frac{m_0 l}{4\pi^2} = 0.10866 m_0 l$$

12-20 Cont:

$$\begin{aligned}
 m_{22} &= m_0 \int_0^l \left(\frac{x}{l}\right) \left(1 - \frac{x}{l}\right) \sin^2 \frac{2\pi x}{l} dx = m_0 \int_0^l \frac{x}{l} \left(1 - \frac{x}{l}\right) \frac{1}{2} (1 - \cos \frac{4\pi x}{l}) dx \\
 &= \frac{m_0 l}{12} - \frac{m_0}{2} \int_0^l \frac{x}{l} \left(1 - \frac{x}{l}\right) \cos \frac{4\pi x}{l} dx = \frac{m_0 l}{12} + \frac{m_0 l}{(4\pi)^2} \\
 &= 0.08966 m_0 l
 \end{aligned}$$

$$m_{12} = m_0 \int_0^l \left(\frac{x}{l}\right) \left(1 - \frac{x}{l}\right) \sin \frac{\pi x}{l} \sin^2 \frac{2\pi x}{l} dx = 0 \text{ by inspection}$$

ie $\frac{x}{l} \left(1 - \frac{x}{l}\right)$ = symmetric function

$\sin \frac{\pi x}{l} \sin^2 \frac{2\pi x}{l}$ = unsymmetric function

Eq. (12-3-7) becomes

$$\begin{bmatrix} \left[EI \left(\frac{\pi}{l}\right)^4 \frac{l}{2} - 0.10866 m_0 l \omega^2 \right] & 0 \\ 0 & \left[EI \left(\frac{2\pi}{l}\right)^4 \frac{l}{2} - 0.08966 m_0 l \omega^2 \right] \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

$\therefore C_1$ & C_2 are independent & Rayleigh-Ritz method fails.

However from above

$$\left[EI \left(\frac{\pi}{l}\right)^4 \frac{l}{2} - 0.10866 m_0 l \omega^2 \right] = 0 \quad \left[EI \left(\frac{2\pi}{l}\right)^4 \frac{l}{2} - 0.08966 m_0 l \omega^2 \right] = 0$$

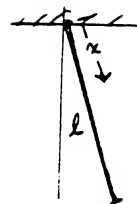
reduces to two Rayleigh's method.

$$\omega_1^2 = \frac{\pi^4}{2 \times 0.10866} \left(\frac{EI}{m_0 l^4} \right) = 448 \frac{EI}{m_0 l^4} \quad \omega_1 = 21.2 \sqrt{\frac{EI}{m_0 l^4}}$$

$$\omega_2^2 = \frac{8\pi^4}{0.08966} \left(\frac{EI}{m_0 l^4} \right) = 8691 \frac{EI}{m_0 l^4} \quad \omega_2 = 93 \sqrt{\frac{EI}{m_0 l^4}}$$

1st mode is $\sin \frac{\pi x}{l}$, 2nd mode is $\sin \frac{2\pi x}{l}$. Compare with Prob 12-11. (ω , agrees)

12-21



$$\phi_1 = \frac{x}{l} \quad \phi_1'' = 0$$

$$\phi_2 = \sin \frac{\pi x}{l} \quad \phi_2'' = -\left(\frac{\pi}{l}\right)^2 \sin \frac{\pi x}{l}$$

$$\phi_3 = \sin \frac{2\pi x}{l} \quad \phi_3'' = -\left(\frac{2\pi}{l}\right)^2 \sin \frac{2\pi x}{l}$$

$$k_{11} = EI \int_0^l \phi_1'' \phi_1'' dx = 0 \quad k_{12} = k_{13} = 0$$

$$k_{22} = EI \left(\frac{\pi}{l}\right)^4 \int_0^l \sin^2 \frac{\pi x}{l} dx = EI \left(\frac{\pi}{l}\right)^4 \frac{l}{2}$$

$$k_{33} = EI \left(\frac{2\pi}{l}\right)^4 \frac{l}{2}$$

$$k_{23} = EI \left(\frac{\pi}{l}\right)^2 \left(\frac{2\pi}{l}\right)^2 \int_0^l \sin \frac{\pi x}{l} \cdot \sin \frac{2\pi x}{l} dx = 0$$

$$m_{11} = m_0 \int_0^l \phi_1^2 dx = m_0 \int_0^l \left(\frac{x}{l}\right)^2 dx = \frac{m_0 l}{3}$$

$$m_{22} = m_0 \int_0^l \sin^2 \frac{\pi x}{l} dx = \frac{m_0 l}{2}$$

$$m_{33} = m_0 \int_0^l \sin^2 \frac{2\pi x}{l} dx = \frac{m_0 l}{2}$$

$$m_{12} = m_0 \int_0^l \frac{x}{l} \sin \frac{\pi x}{l} dx = \frac{m_0}{l} \left[\frac{\sin \frac{\pi x}{l}}{\left(\frac{\pi}{l}\right)^2} - x \frac{\cos \frac{\pi x}{l}}{\left(\frac{\pi}{l}\right)} \right]_0^l$$

$$= \frac{m_0 l}{\pi} = m_{21}$$

$$m_{13} = m_0 \int_0^l \frac{x}{l} \sin \frac{2\pi x}{l} dx = -\frac{m_0 l}{2\pi}$$

12-21 Cont

$$m_{23} = m_0 \int_0^l \sin \frac{\pi x}{l} \sin \frac{2\pi x}{l} dx = 0$$

$$\begin{bmatrix} (0 - \omega^2 \frac{m_0 l}{3}) & (0 - \omega^2 \frac{m_0 l}{\pi}) & (0 + \omega^2 \frac{m_0 l}{2\pi}) \\ (0 - \omega^2 \frac{m_0 l}{\pi}) & (EI(\frac{\pi}{l})^4 \frac{l}{2} - \omega^2 \frac{m_0 l}{2}) & (0 - 0) \\ (0 - \omega^2 \frac{m_0 l}{2\pi}) & (0 - 0) & (EI(\frac{2\pi}{l})^4 \frac{l}{2} - \omega^2 \frac{m_0 l}{2}) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Freq. Eq.

$$\begin{aligned} & -\omega^2 \frac{m_0 l}{3} [EI(\frac{\pi}{l})^4 \frac{l}{2} - \omega^2 \frac{m_0 l}{2}] [EI(\frac{2\pi}{l})^4 \frac{l}{2} - \omega^2 \frac{m_0 l}{2}] \\ & + \omega^2 \frac{m_0 l}{\pi} \left[-(\omega^2 \frac{m_0 l}{\pi}) \left\{ EI(\frac{\pi}{l})^4 \frac{l}{2} - \omega^2 \frac{m_0 l}{2} \right\} \right] \\ & - \omega^2 \frac{m_0 l}{2\pi} \left[-(\omega^2 \frac{m_0 l}{2\pi}) \left\{ EI(\frac{\pi}{l})^4 \frac{l}{2} - \omega^2 \frac{m_0 l}{2} \right\} \right] = 0 \end{aligned}$$

$$\begin{aligned} & -(\omega^2 m_0 l)^3 \left[\frac{1}{12} - \frac{1}{2\pi^2} + \frac{1}{8\pi^2} \right] + (\omega^2 m_0 l)^2 EI \left[\frac{l}{12} \left(\frac{\pi}{l} \right)^4 + \frac{l}{12} \left(\frac{2\pi}{l} \right)^4 - \frac{l}{2\pi^2} \left(\frac{\pi}{l} \right)^4 + \frac{l}{8\pi^2} \left(\frac{2\pi}{l} \right)^4 \right] \\ & - (\omega^2 m_0 l) (EI)^2 \frac{l^2}{12} \left(\frac{\pi}{l} \right)^4 \left(\frac{2\pi}{l} \right)^4 = 0 \end{aligned}$$

$$-0.045338(\omega^2)^3 + 60.273(\omega^2)^2 \left(\frac{EI}{m_0 l^4} \right) - 12651.37 \omega^2 \left(\frac{EI}{m_0 l^4} \right)^2 = 0$$

$$(\omega^2)^2 - (1329.4146 \frac{EI}{m_0 l^4}) \omega^2 + 279045.61 \left(\frac{EI}{m_0 l^4} \right)^2 = 0$$

$$\omega^2 = (664.70 \pm 403.47) \left(\frac{EI}{m_0 l^4} \right) = \begin{Bmatrix} 261.23 \\ 1068.17 \end{Bmatrix} \frac{EI}{m_0 l^4}$$

$$\omega = \begin{Bmatrix} 0.0 \\ 16.16 \\ 32.68 \end{Bmatrix} \sqrt{\frac{EI}{m_0 l^4}} \quad \text{Exact Sol } \omega = \begin{Bmatrix} 0 \\ 15.4 \\ 50. \end{Bmatrix} \sqrt{\frac{EI}{m_0 l^4}}$$

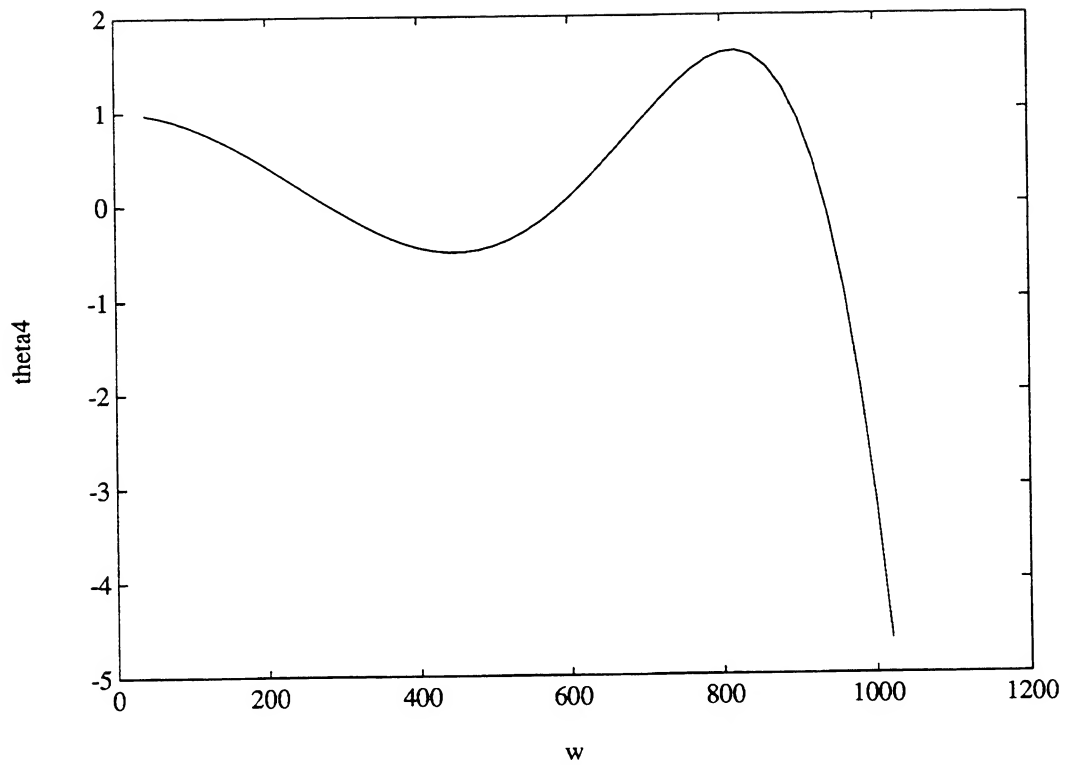
12-22

use the program for.m

$$J = [1, 1, 2, 0]$$

$$K = [.2, .4, .6] \times 10^6$$

$$I = 1.5 \times 10^{-5}$$



12-24 cont

>>[w',theta4']

ans =

1.0e+03 *

0.0400	0.0010
0.0600	0.0009
0.0800	0.0009
0.1000	0.0008
0.1200	0.0008
0.1400	0.0007
0.1600	0.0006
0.1800	0.0005
0.2000	0.0004
0.2200	0.0003
0.2400	0.0002
0.2600	0.0001
0.2800	0.0000
0.3000	-0.0001
0.3200	-0.0002
0.3400	-0.0003
0.3600	-0.0003
0.3800	-0.0004
0.4000	-0.0005
0.4200	-0.0005
0.4400	-0.0005
0.4600	-0.0005
0.4800	-0.0005
0.5000	-0.0004
0.5200	-0.0004
0.5400	-0.0003
0.5600	-0.0002
0.5800	-0.0001
0.6000	0.0001
0.6200	0.0002
0.6400	0.0004
0.6600	0.0006
0.6800	0.0008
0.7000	0.0009
0.7200	0.0011
0.7400	0.0013
0.7600	0.0014
0.7800	0.0015
0.8000	0.0016

← 1st mode

← 2nd mode

w' theta4

1.0e+03*

0.8200	0.0016
0.8400	0.0016
0.8600	0.0014
0.8800	0.0012
0.9000	0.0009
0.9200	0.0004
0.9400	-0.0002
0.9600	-0.0010
0.9800	-0.0020
1.0000	-0.0032
1.0200	-0.0046

← 3rd mode

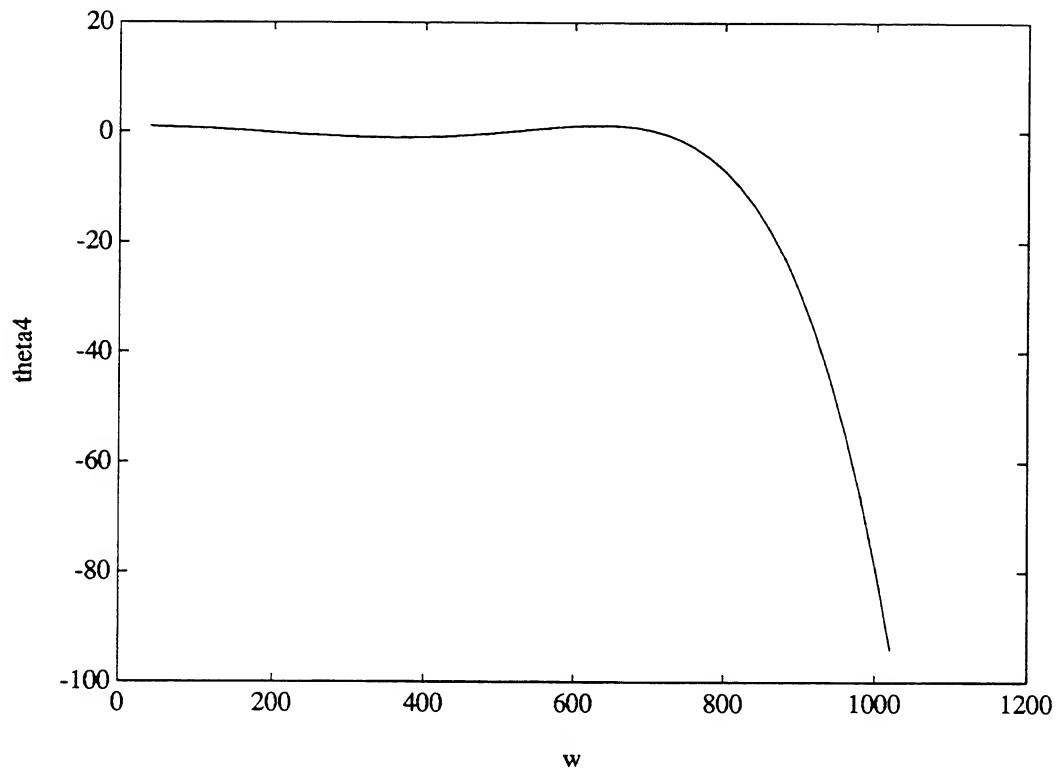
12-23

Use Tor.m

$$J = [1.13, 1.13, 1.13, 2.26]$$

$$K = [.169, .169, .224] \times 10^6$$

$$\Gamma = 1.5 \times 10^{-5}$$



12-23 corr.

>>>[w',theta4']

ans =

1.0e+03 *

0.0400	0.0009
0.0600	0.0009
0.0800	0.0008
0.1000	0.0007
0.1200	0.0005
0.1400	0.0004
0.1600	0.0002
0.1800	0.0000
0.2000	-0.0001
0.2200	-0.0003
0.2400	-0.0005
0.2600	-0.0006
0.2800	-0.0008
0.3000	-0.0009
0.3200	-0.0010
0.3400	-0.0010
0.3600	-0.0010
0.3800	-0.0010
0.4000	-0.0010
0.4200	-0.0009
0.4400	-0.0007
0.4600	-0.0005
0.4800	-0.0003
0.5000	-0.0001
0.5200	0.0001
0.5400	0.0004
0.5600	0.0007
0.5800	0.0009
0.6000	0.0011
0.6200	0.0012
0.6400	0.0012
0.6600	0.0011
0.6800	0.0008
0.7000	0.0004
0.7200	-0.0004
0.7400	-0.0014
0.7600	-0.0028
0.7800	-0.0046
0.8000	-0.0070

← 1st mode

← 2nd mode

← 3rd mode

w' theta4'

1.0 E+03 *

0.8200	-0.0099
0.8400	-0.0134
0.8600	-0.0178
0.8800	-0.0230
0.9000	-0.0291
0.9200	-0.0364
0.9400	-0.0449
0.9600	-0.0547
0.9800	-0.0661
1.0000	-0.0792
1.0200	-0.0941

12-24

%This is the Holzer method for the spring mass system

clear; clg;

m=[1 1 1];

k=[1 1 1];

for I=1:13

x(1)=1;

w(I)=(I-1)*.2;

lambda(I)=(w(I))^2;

x(2)=1- lambda(I)*m(1)/k(1);

f(1)=lambda(I)*m(1)*x(1);

ff=f(1);

term=1;

for ii=2:3

term=term+x(ii);

f(ii)=lambda(I)*term;

ff=ff+f(ii);

x(ii+1)=x(ii)-ff/k(ii);

end

x4(I)=x(4);

end

plot(w,x4)

[w',x4']

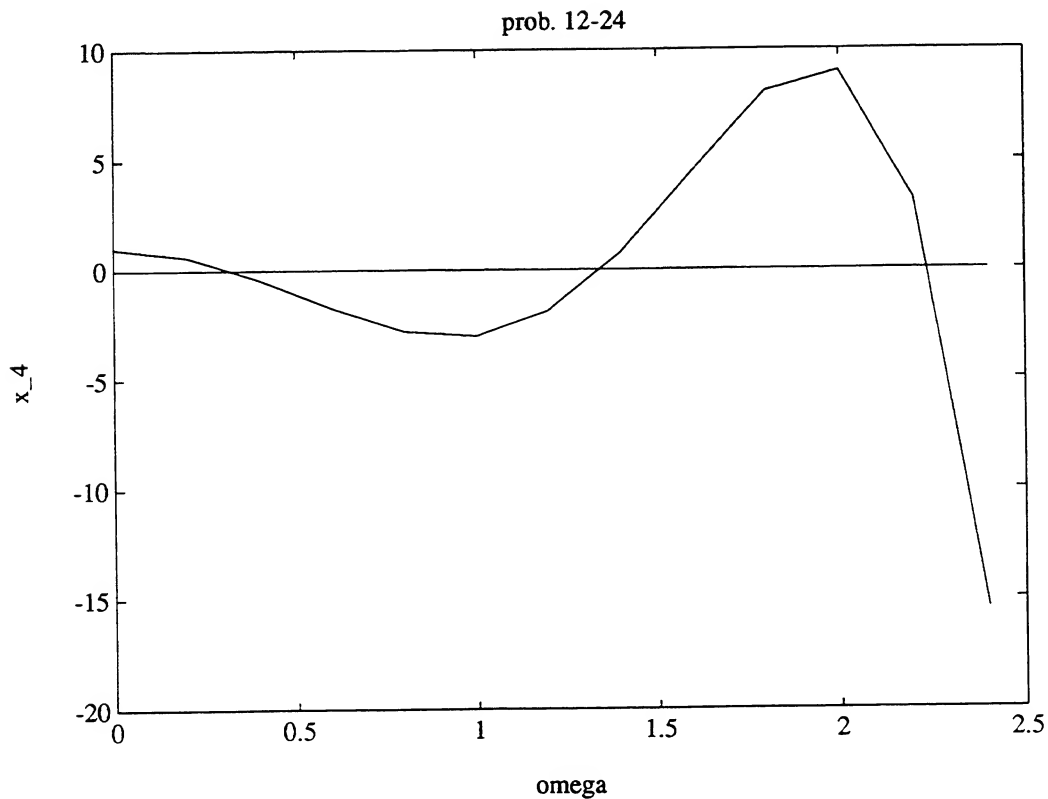
ans =

0	1.0000
0.2000	0.6111
0.4000	-0.4249
0.6000	-1.7395
0.8000	-2.7949
1.0000	-3.0000
1.2000	-1.8708
1.4000	0.7617
1.6000	4.4980
1.8000	8.0710
2.0000	9.0000
2.2000	3.1993
2.4000	-15.4598

← ω_1

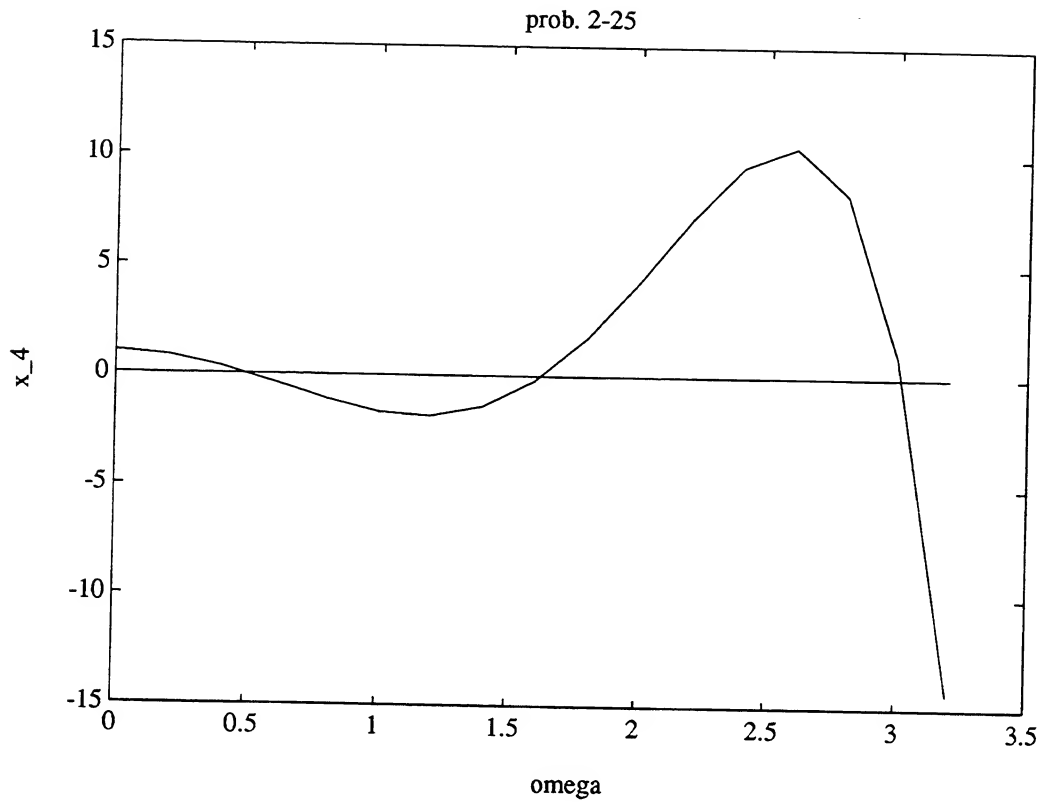
← ω_2

← ω_3



12-25

use the same program as 12-24



[w', x4']

ans =

0	1.0000
0.2000	0.8232
0.4000	0.3305
0.6000	-0.3686
0.8000	-1.1045
1.0000	-1.6667
1.2000	-1.8305
1.4000	-1.3917
1.6000	-0.2090
1.8000	1.7465
2.0000	4.3333
2.2000	7.1745
2.4000	9.5847
2.6000	10.4892
2.8000	8.3361
3.0000	1.0000
3.2000	-14.3218

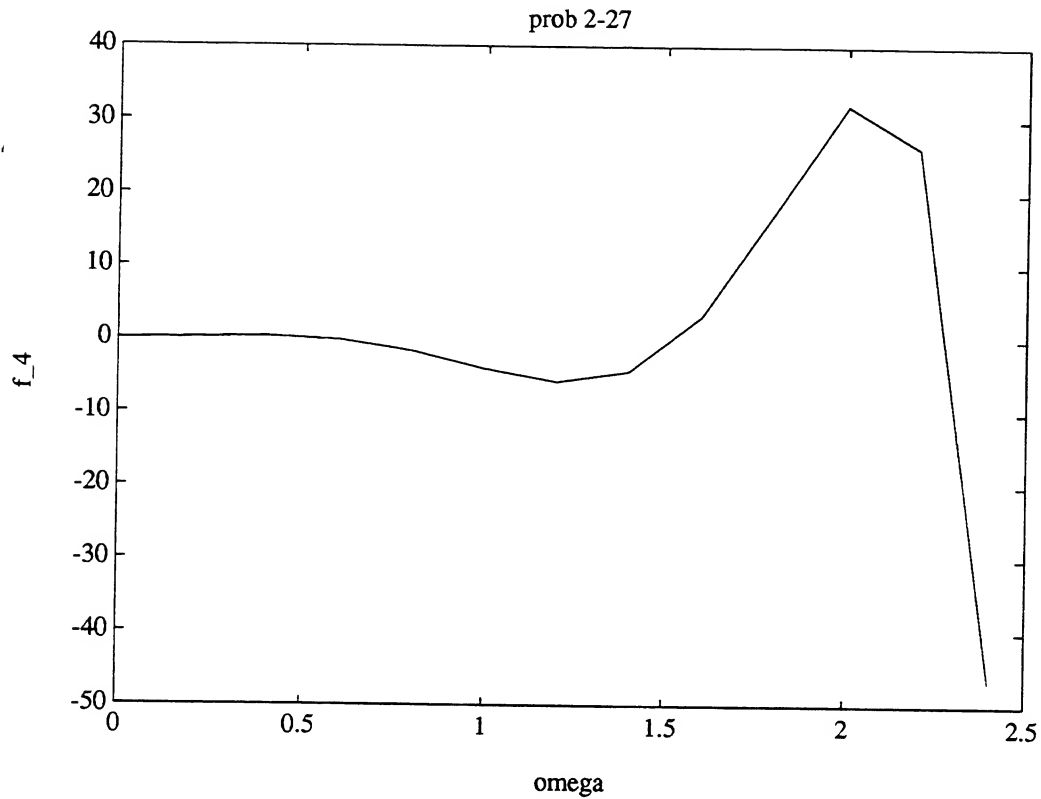
$\leftarrow \omega_1$

$\leftarrow \omega_2$

$\leftarrow \omega_3$

12-27

Use the same program as 12-25
Extend ii index to 4.



[w' f4']

ans =

0	0
0.2000	0.1365
0.4000	0.2881
0.6000	-0.1475
0.8000	-1.6546
1.0000	-4.0000
1.2000	-5.7559
1.4000	-4.3056
1.6000	3.2041
1.8000	17.3942
2.0000	32.0000
2.2000	26.2565
2.4000	-46.5533

ω_1 (pointing to 0.6000, -0.1475)
 ω_2 (pointing to 1.6000, 3.2041)
 ω_3 (pointing to 2.4000, -46.5533)

12-28 Use the program for.m=

$$J = [50, 138, 145, 181, 240, 1/2(140000)]$$

$$K = 10^6 * [15, 30, 22, 36, 120, 0]$$

The system was cut at centerline using $1/2$ the J of the fuselage. Antisymmetric modes are given for $\theta_6 = 0$ and symmetric modes for $T_6 = 0$. Note: they are very close to each other because of the very large J_6

»[w' theta6' t6']

ans =

1.0e+10 *

0	0.00000000010000	0
0.00000000100000	0.00000000009955	0.00070454393355
0.00000000200000	0.00000000009819	0.00277996177187
0.00000000300000	0.00000000009594	0.00611264965261
0.00000000400000	0.00000000009283	0.01051670778083
0.00000000500000	0.00000000008889	0.01573903028613
0.00000000600000	0.00000000008416	0.02146629176526
0.00000000700000	0.00000000007870	0.02733369347504
0.00000000800000	0.00000000007256	0.03293529614420
0.00000000900000	0.00000000006581	0.03783573271471
0.00000001000000	0.00000000005852	0.04158306348715
0.00000001100000	0.00000000005078	0.04372250859723
0.00000001200000	0.00000000004267	0.04381076893278
0.00000001300000	0.00000000003428	0.04143062692522
0.00000001400000	0.00000000002570	0.03620550350001
0.00000001500000	0.00000000001702	0.02781363719313
0.00000001600000	0.00000000000834	0.01600154634395
0.00000001700000	-0.00000000000023	0.00059643562529
0.00000001800000	-0.00000000000862	-0.01848278581237
0.00000001900000	-0.00000000001672	-0.04121619545193
0.00000002000000	-0.00000000002446	-0.06747556083990
0.00000002100000	-0.00000000003173	-0.09701933458121
0.00000002200000	-0.00000000003846	-0.12949019031575
0.00000002300000	-0.00000000004459	-0.16441531332092
0.00000002400000	-0.00000000005004	-0.20120961780241

12-28 cont.

ω	Θ_{ψ}	T_{ψ}
0.00000002500000	-0.00000000005475	-0.23918201574982
0.00000002600000	-0.00000000005869	-0.27754480972827
0.00000002700000	-0.00000000006180	-0.31542622451125
0.00000002800000	-0.00000000006406	-0.35188603048073
0.00000002900000	-0.00000000006546	-0.38593414575746
0.00000003000000	-0.00000000006598	-0.41655203470013
0.00000003100000	-0.00000000006564	-0.44271664844260
0.00000003200000	-0.00000000006445	-0.46342657933742
0.00000003300000	-0.00000000006244	-0.47773002645629
0.00000003400000	-0.00000000005966	-0.48475409468513
0.00000003500000	-0.00000000005615	-0.48373487657210
0.00000003600000	-0.00000000005197	-0.47404769518270
0.00000003700000	-0.00000000004721	-0.45523681914657
0.00000003800000	-0.00000000004195	-0.42704380022221
0.00000003900000	-0.00000000003628	-0.38943432166274
0.00000004000000	-0.00000000003030	-0.34262062466385
0.00000004100000	-0.00000000002411	-0.28708209408002
0.00000004200000	-0.00000000001783	-0.22357962439411
0.00000004300000	-0.00000000001157	-0.15316492796073
0.00000004400000	-0.00000000000545	-0.07718318108356
0.00000004500000	0.00000000000042	0.00273177604275
0.00000004600000	0.00000000000592	0.08467051902641
0.00000004700000	0.00000000001094	0.16647015791455
0.00000004800000	0.00000000001539	0.24574615238018
0.00000004900000	0.00000000001916	0.31993434882400
0.00000005000000	0.00000000002218	0.38634359875886
0.00000005100000	0.00000000002436	0.44221911472193
0.00000005200000	0.00000000002566	0.48481647616685
0.00000005300000	0.00000000002603	0.51148590960207
0.00000005400000	0.00000000002546	0.51976613076509
0.00000005500000	0.00000000002394	0.50748664777117
0.00000005600000	0.00000000002149	0.47287697868056
0.00000005700000	0.00000000001817	0.41468073033284
0.00000005800000	0.00000000001403	0.33227191294532
0.00000005900000	0.00000000000918	0.22577022201726
0.00000006000000	0.00000000000372	0.09615130046836
0.00000006100000	-0.00000000000219	-0.05465280558446
0.00000006200000	-0.00000000000840	-0.22366866991186
0.00000006300000	-0.00000000001472	-0.40681863857555
0.00000006400000	-0.00000000002095	-0.59888783801372
0.00000006500000	-0.00000000002688	-0.79352994513813
0.00000006600000	-0.00000000003227	-0.98332191613568
0.00000006700000	-0.00000000003690	-1.15987927799034
0.00000006800000	-0.00000000004056	-1.31404511781761
0.00000006900000	-0.00000000004302	-1.43616756696207

12-20 cont.

ω	θ_0	T_0
0.00000007000000	-0.00000000004411	-1.51648237671449
0.00000007100000	-0.00000000004366	-1.54561912796732
0.00000007200000	-0.00000000004159	-1.51525171589879
0.00000007300000	-0.00000000003784	-1.41891601085994
0.00000007400000	-0.00000000003247	-1.25302002629042
0.00000007500000	-0.00000000002563	-1.01807453222036
0.00000007600000	-0.00000000001759	-0.72017484749937
0.00000007700000	-0.00000000000878	-0.37276753436557
0.00000007800000	0.00000000000019	0.00126108536749
0.00000007900000	0.00000000000849	0.36713426279700
0.00000008000000	0.00000000001506	0.67532803833047
0.00000008100000	0.00000000001853	0.85826050744362
0.00000008200000	0.00000000001722	0.82647518628648
0.00000008300000	0.00000000000905	0.46426215681840
0.00000008400000	-0.00000000000848	-0.37534396649941
0.00000008500000	-0.00000000003835	-1.87625512220048

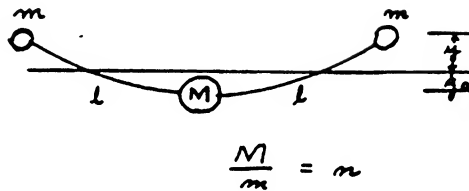
12-30

From conservation of
momentum

$$M y_0 = 2m y$$

$$(y + y_0) = \frac{Pl^3}{3EI} = \frac{(m\omega^2 y)l^3}{3EI} = y + \frac{2m}{M} y = y \left(1 + \frac{2}{n}\right)$$

$$\therefore \omega^2 = \frac{3EI}{Ml^3} (n+2) = \frac{6EI}{Ml^3} \left(1 + \frac{n}{2}\right) \quad \omega = \sqrt{\frac{6EI}{Ml^3} \left(1 + \frac{n}{2}\right)}$$



~~14.21~~

Using Eq. 12.12-5

$$\begin{Bmatrix} -V_3 \\ M_3 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 \\ l & 1 & 0 & m\omega^2 l \\ 3\alpha l & 6\alpha & 1 & m\omega^2 3\alpha l \\ \alpha l^2 & 3\alpha l & l & (1+m\omega^2 \alpha l^2) \end{bmatrix}^2 \begin{Bmatrix} -0 \\ 0 \\ 1 \\ \theta_1 \end{Bmatrix} \quad \text{where} \quad \alpha = \frac{l}{6EI}$$

Calculation can be limited to last 2 rows of last two columns.

$$\left. \begin{aligned} u_{33} + u_{34} \theta_1 &= 0 \\ u_{43} + u_{44} \theta_1 &= 0 \end{aligned} \right\} \text{ or } \frac{u_{33} u_{44} - u_{34} u_{43}}{\text{freq. eq.}} = 0$$

$$\text{For } A^2 = A A = U, \quad u_{ij} = \sum_k a_{ik} a_{kj}$$

$$\therefore u_{33} = \sum_k a_{3k} a_{k3} = 1 + m\omega^2 3\alpha l^2$$

$$\begin{aligned} u_{44} &= \sum_k a_{4k} a_{k4} = m\omega^2 \alpha l^2 + m\omega^2 3\alpha l^2 + m\omega^2 3\alpha l^2 + (1+m\omega^2 \alpha l^2)^2 \\ &= 1 + 9m\omega^2 \alpha l^2 + (m\omega^2 \alpha l^2)^2 \end{aligned}$$

$$\begin{aligned} u_{34} &= \sum_k a_{3k} a_{k4} = m\omega^2 3\alpha l + m\omega^2 6\alpha l + m\omega^2 3\alpha l + m\omega^2 3\alpha l (1+m\omega^2 \alpha l^2) \\ &= 15m\omega^2 \alpha l + 3(m\omega^2 \alpha l^2)^2 \end{aligned}$$

$$u_{43} = l + l(1+m\omega^2 \alpha l^2) = 2l + m\omega^2 \alpha l^3$$

subst. into freq. eq.

$$\begin{aligned} & (1+m\omega^2 3\alpha l^2) [1+9m\omega^2 \alpha l^2 + (m\omega^2 \alpha l^2)^2] \\ & + (2l + m\omega^2 \alpha l^3) [-15m\omega^2 \alpha l - 3l(m\omega^2 \alpha l^2)^2] = 0 \end{aligned}$$

$$\text{or } 1 - 18(m\omega^2 \alpha l^2) + 7(m\omega^2 \alpha l^2)^2 = 0$$

$$\text{let } \beta = m\omega^2 \alpha l^2 \quad \text{then} \quad \beta^2 - \frac{18}{7} \beta + \frac{1}{7} = 0$$

$$\beta = 1.2857 \pm \sqrt{1.6531 - .1429} = 1.2857 \pm 1.2289$$

$$\beta = \begin{cases} .0568 \\ 2.5615 \end{cases} \quad \sqrt{\beta} = \begin{cases} .2384 \\ 1.5857 \end{cases} \quad \omega = \begin{cases} .584 \\ 3.884 \end{cases} \sqrt{\frac{EI}{ml^3}}$$

%This program uses Myklestad's method on a three mass beam. It is
 % set up for evenly spaced lumped masses.

```
clear; clg;
n=3;
mass=[1 1 1];
l=1;
ei=.1e7;
coef=l/ei;
coef2=(l^2)/(2*ei);
coef3=(l^3)/(3*ei);
for j=1:200
omega(j)=150 +10*j;
va(1)=0;
ma(1)=0;
thetaa(1)=0;
ya(1)=1.;
for i=1:n
    va(i+1)=va(i)-mass(i)*omega(j)^2*ya(i);
    ma(i+1)=ma(i)-va(i+1)*l;
    thetaa(i+1)=thetaa(i)+ma(i+1)*coef+va(i+1)*coef2;
    ya(i+1)=ya(i)+l*thetaa(i)+ma(i+1)*coef2+va(i+1)*coef3;
end
vb(1)=0;
mb(1)=0;
thetab(1)=1;
yb(1)=0;
for i=1:n
    vb(i+1)=vb(i)-mass(i)*omega(j)^2*yb(i);
    mb(i+1)=mb(i)-vb(i+1)*l;
    thetab(i+1)=thetab(i)+mb(i+1)*coef+vb(i+1)*coef2;
    yb(i+1)=yb(i)+l*thetab(i)+mb(i+1)*coef2+vb(i+1)*coef3;
end
thetal=-thetaa(4)/thetab(4);
y4(j)=ya(4)+yb(4)*thetal;
end
[omega' y4']
```

12-31 cont.

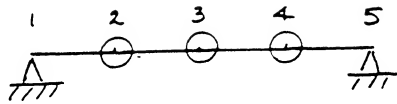
Ω
1.0e+03 * γ

0.1600	0.0006
0.1700	0.0006
0.1800	0.0006
0.1900	0.0005
0.2000	0.0005
0.2100	0.0004
0.2200	0.0004
0.2300	0.0003
0.2400	0.0003
0.2500	0.0002
0.2600	0.0002
0.2700	0.0001
0.2800	0.0001
0.2900	0.0000 $\leftarrow \omega_1$
0.3000	-0.0000
0.3100	-0.0001
0.3200	-0.0001
0.3300	-0.0002
0.3400	-0.0003
0.3500	-0.0003
0.3600	-0.0004
0.3700	-0.0004
0.3800	-0.0005
0.3900	-0.0005
0.4000	-0.0006
0.4100	-0.0006
0.4200	-0.0007
0.4300	-0.0007
0.4400	-0.0007

values
deleted
 \leftarrow

Ω	γ
1.7200	-0.0005
1.7300	-0.0005
1.7400	-0.0005
1.7500	-0.0005
1.7600	-0.0004
1.7700	-0.0004
1.7800	-0.0004
1.7900	-0.0004
1.8000	-0.0003
1.8100	-0.0003
1.8200	-0.0003
1.8300	-0.0002
1.8400	-0.0002
1.8500	-0.0002
1.8600	-0.0002
1.8700	-0.0001
1.8800	-0.0001
1.8900	-0.0001
1.9000	-0.0000
1.9100	-0.0000 $\leftarrow \Omega_2$
1.9200	0.0000
1.9300	0.0000
1.9400	0.0001
1.9500	0.0001
1.9600	0.0001
1.9700	0.0002
1.9800	0.0002

12-32



Stations must be numbered as 1 to 5 with
 $m_1 = m_5 = 0$

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_5 = \begin{bmatrix} - & - & - & - \\ u_{21} & - & u_{23} & - \\ - & - & - & - \\ u_{41} & - & u_{43} & - \end{bmatrix} \begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_1$$

$$\therefore \begin{vmatrix} u_{21} & u_{23} \\ u_{41} & u_{43} \end{vmatrix} = 0 \quad \text{or} \quad u_{43} = \frac{u_{41} u_{23}}{u_{21}}$$

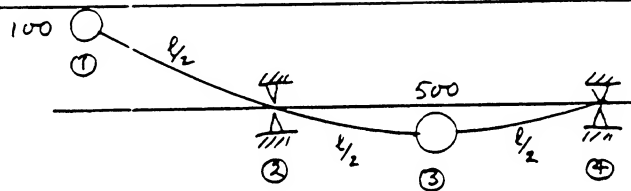
12-33

$$m_2 = m_4 = 0$$

$$y_1 = 1.0$$

$$y_2 = y_4 = 0$$

Eq. 1, 1-5



$$\begin{Bmatrix} -V \\ M \\ \theta \\ 0 \end{Bmatrix}_2 = \begin{bmatrix} \text{sec 1} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1 \quad \begin{Bmatrix} -V \\ M \\ \theta \\ y \end{Bmatrix}_3 = \begin{bmatrix} \text{sec 2} \end{bmatrix} \begin{Bmatrix} -V \\ M \\ \theta \\ 0 \end{Bmatrix}_2 \quad \text{etc}$$

this is constraint eq.

$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_4 = \begin{bmatrix} 0 & 0 \\ \text{sec 1} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \text{sec 2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1 = \begin{bmatrix} - & - & - & - \\ - & - & u_{23} & u_{24} \\ - & - & - & - \\ - & - & u_{43} & u_{44} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \end{Bmatrix}_1$$

$$\begin{vmatrix} u_{23} & u_{24} \\ u_{43} & u_{44} \end{vmatrix} = 0$$

12-34

Eqs. 12-1 to 6 can be arranged in the following matrix form

$$\begin{Bmatrix} -V \\ M \\ \theta \\ y \\ T \\ \varphi \end{Bmatrix}_{i+1} = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 & 0 & m\omega^2 c \\ l & 1 & 0 & m\omega^2 l & 0 & m\omega^2 cl \\ \frac{l^2}{2EI} & \frac{l}{EI} & 1 & \frac{m\omega^2 l^2}{2EI} & 0 & \frac{m\omega^2 cl^2}{2EI} \\ \frac{l^2}{6EI} & \frac{l}{2EI} & l & (1 + \frac{m\omega^2 l^3}{6EI}) & 0 & \frac{m\omega^2 cl^3}{6EI} \\ 0 & 0 & 0 & m\omega^2 c & 1 & J\omega^2 \\ 0 & 0 & 0 & m\omega^2 cl & l & (1 + J\omega^2 l) \end{bmatrix} \begin{Bmatrix} -V \\ M \\ \theta \\ y \\ T \\ \varphi \end{Bmatrix}_i$$

$$\begin{Bmatrix} V \\ W \end{Bmatrix}_3 = \begin{bmatrix} A & 2B \\ 2C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} V \\ W \end{Bmatrix}_1$$

$$= \begin{bmatrix} (A^2 + 2BC) & (AB + 2BD) \\ (2AC + CD) & (2BC + D^2) \end{bmatrix} \begin{Bmatrix} V \\ W \end{Bmatrix}_1$$

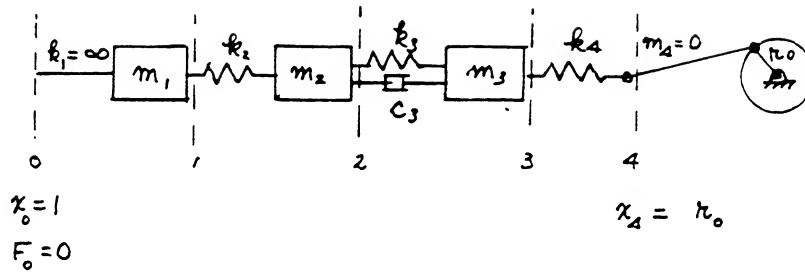
Section 2 differs from Sec 1 by $2C_i$ in place of C_i .

$$\begin{Bmatrix} -V_3 \\ M_3 \\ 0 \\ 0 \\ T_3 \\ 0 \end{Bmatrix} = \begin{bmatrix} u_{ij} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta_1 \\ 1 \\ 0 \\ \varphi_1 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} u_{33} & u_{34} & u_{36} \\ u_{43} & u_{44} & u_{46} \\ u_{63} & u_{64} & u_{66} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ 1 \\ \varphi_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Set determinant to zero for nat. freqs.

12-35



$$\begin{Bmatrix} x \\ F \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ -m_1 \omega^2 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{\infty} \\ -m_1 \omega^2 & 1 - \frac{m_1 \omega^2}{\infty} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}_0 = \begin{Bmatrix} 1 \\ -m_1 \omega^2 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_2 = \begin{bmatrix} 1 & \frac{1}{k_2} \\ -m_2 \omega^2 & (1 - \frac{m_2 \omega^2}{k_2}) \end{bmatrix} \begin{Bmatrix} x \\ F \end{Bmatrix}_1$$

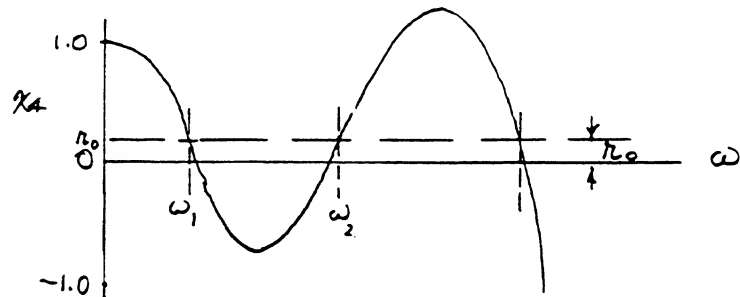
$$\begin{Bmatrix} x \\ F \end{Bmatrix}_3 = \begin{bmatrix} 1 & \frac{1}{k_3 + i\omega C_3} \\ -m_3 \omega^2 & (1 - \frac{m_3 \omega^2}{k_3 + i\omega C_3}) \end{bmatrix} \begin{Bmatrix} x \\ F \end{Bmatrix}_2$$

$$\begin{Bmatrix} x \\ F \end{Bmatrix}_4 = \begin{bmatrix} 1 & \frac{1}{k_4} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ F \end{Bmatrix}_3$$

Boundary Cond.

$$\therefore x_4 = x_3 + \frac{F_3}{k_4} = r_0$$

Problem is identical in calculation to torsional system and same program as Prob 12-3 can be used.



12-36

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0^R = \begin{Bmatrix} 1 \\ -\omega^2 J_1 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_m^R = \begin{bmatrix} 1 & 0 \\ -\omega^2 J & 1 \end{bmatrix}_m \begin{bmatrix} 1 & \frac{1}{k+i\omega g} \\ 0 & 1 \end{bmatrix}_m \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{m-1}^R = \begin{bmatrix} 1 & \frac{1}{k+i\omega g} \\ -\omega^2 J & (1 - \frac{\omega^2 J}{k+i\omega g}) \end{bmatrix}_m \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{m-1}^R$$

Elements of above transfer matrix are:

$$\frac{1}{k+i\omega g} = \left[\frac{k}{k^2 + (\omega g)^2} \right] - i \left[\frac{\omega g}{k^2 + (\omega g)^2} \right]$$

$$1 - \frac{\omega^2 J}{k+i\omega g} = \left[1 - \frac{\omega^2 J k}{k^2 + (\omega g)^2} \right] + i \left[\frac{\omega^2 J \omega g}{k^2 + (\omega g)^2} \right]$$

Sample calc. for $\omega^2 = 0.5 \times 10^5$

Renumber J_s from J_0 to J_3

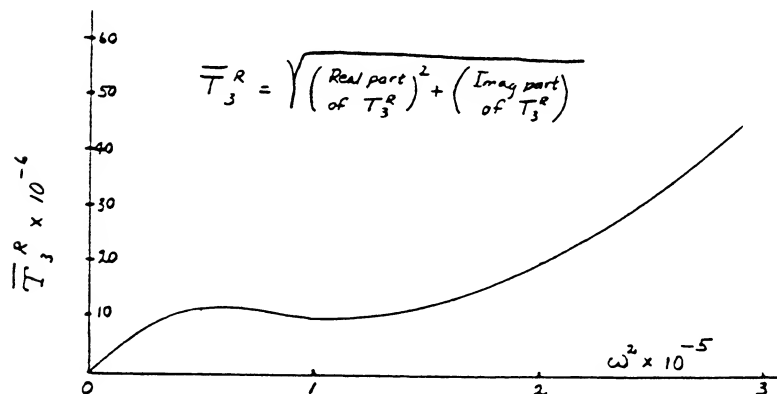
$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0^R = \begin{Bmatrix} 1.0 \\ -1 \times 10^6 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_1^R = \begin{bmatrix} 1 & -i(4.46 \times 10^{-6}) \\ -5 \times 10^6 & (1 + i2.23) \end{bmatrix} \begin{Bmatrix} 1.0 \\ -1 \times 10^6 \end{Bmatrix} = \begin{Bmatrix} 1 + .446i \\ (-6 - i2.23) \times 10^6 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2^R = \begin{bmatrix} 1 & .2 \times 10^{-6} \\ -.5 \times 10^6 & .90 \end{bmatrix} \begin{Bmatrix} 1 + .446i \\ (-6 - i2.23) \times 10^6 \end{Bmatrix} = \begin{Bmatrix} -.20 \\ (-5.9 - i2.23) \times 10^6 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_3^R = \begin{bmatrix} 1 & .10 \times 10^{-6} \\ -1.5 \times 10^6 & .85 \end{bmatrix} \begin{Bmatrix} -.20 \\ (-5.9 - i2.23) \times 10^6 \end{Bmatrix} = \begin{Bmatrix} -.79 - i.223 \\ (-4.72 - i1.9) \times 10^6 \end{Bmatrix}$$

Repeat for other frequencies - should be programmed for digital computer.



12-36. cont.

```
%This program is for problem 12-36 and 12-35
clear, clg;
i=sqrt(-1);
m=4;
L=60;
j=[20 100 10 30 ];
k=1e6*[0 0 5 10];
c=[0 0 0 0 ];
g=[0 1 0 0 ]*1e4;
for ii=1:L
w(ii)=ii*10;
for jj=1:4
    zr=-w(ii)^2*j(jj);
    zi=w(ii)*c(jj);
    crj(jj)=zr+i*zi;
    zr=k(jj);
    zi=w(ii)*g(jj);
    crk(jj)=zr+i*zi;
end
de(ii,1)=1;
t(ii,1)=crj(1)*de(ii,1);
for jj=2:m
de(ii,jj)=de(ii,jj-1)+t(ii,jj-1)/crk(jj);
t(ii,jj)=crj(jj)*de(ii,jj-1)+(1+crj(jj)/crk(jj))*t(ii,jj-1);
end
de4(ii)=de(ii,4);
t4(ii)=t(ii,4);
end
```

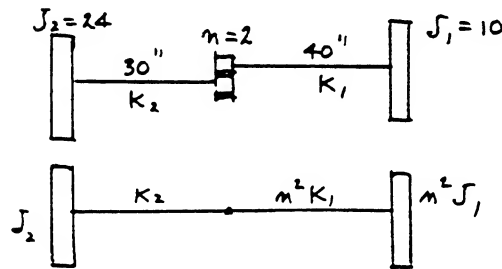
12-37

$$\omega^2 = \frac{(J_2 + m^2 J_1) K_2 (m^2 K_1)}{(m^2 J_1) J_2 (K_2 + m^2 K_1)}$$

$$K_1 = G \frac{\pi D^4}{32 l} = \frac{(12 \times 10^6) \pi (1.5)^4}{32 \times 40} = 0.1492 \times 10^6$$

$$K_2 = \frac{(12 \times 10^6) \pi (2)^4}{32 \times 30} = 0.628 \times 10^6$$

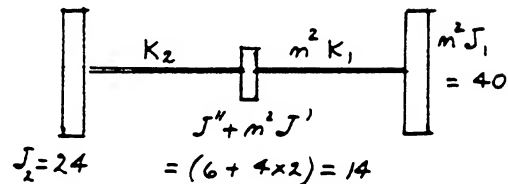
$$\omega^2 = \frac{(24 + 40)(.628)(.1492) \times 10^{12}}{(240)(.628 + .597) \times 10^6} = 2.04 \times 10^4$$



$$\omega = 22.7 \text{ Hz}$$

12-38

After reduction to equivalent single shaft system, the freq. eq. of the degenerate 3-DOF system may be used.

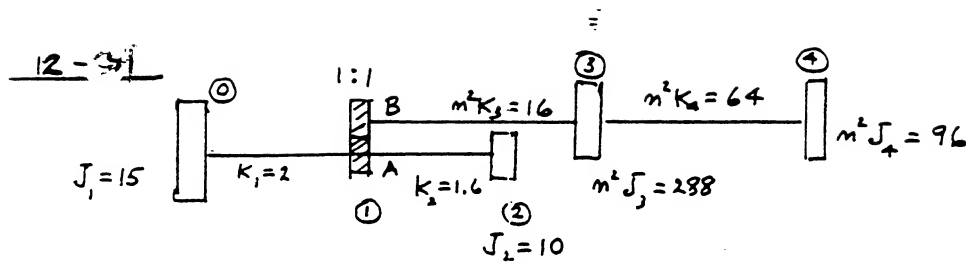


$$\omega^4 - \left[\frac{K_1}{J_1} + \frac{K_2}{J_2} \left(1 + \frac{K_1}{K_2} + \frac{J_2}{J_3} \right) \right] \omega^2 + \frac{K_1 K_2}{J_1 J_2} \left(\frac{J_1 + J_2 + J_3}{J_3} \right) = 0$$

$$\omega^4 - \left[\left(\frac{62.8}{24} \right) + \frac{59.7}{14} \left(1 + \frac{.628}{.597} + \frac{14}{40} \right) \right] 10^4 \omega^2 + \left[\frac{62.8 \cdot 59.7}{24 \cdot 14} \left(\frac{24 + 14 + 40}{40} \right) \right] 10^8 = 0$$

$$\omega^4 - 12.82 \times 10^4 \omega^2 + 21.8 \times 10^6 = 0$$

$$\omega^2 = \begin{cases} 10.86 \times 10^4 \\ 2.01 \times 10^4 \end{cases} \quad \omega = \begin{cases} 329.2 \text{ r/s} \\ 141.8 \text{ " } \end{cases} = \begin{cases} 52.30 \text{ Hz} \\ 22.55 \text{ " } \end{cases}$$



J & K along shaft B are multiplied by $m^2 = 16$ so that the gear ratio is reduced to 1:1

Then $\theta_{A1}^R = -\theta_{B1}^R = \theta_1^L$ and $T_{A1}^R + T_{B1}^R = T_1^L$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_0^R = \begin{Bmatrix} 1 \\ -15\omega^2 \end{Bmatrix} = \text{starting eq.}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_1^L = \begin{bmatrix} 1 & .5 \times 10^{-6} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -15\omega^2 \end{Bmatrix} = \begin{Bmatrix} 1 - 7.5 \times 10^{-6} \omega^2 \\ -15\omega^2 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_3^R = \begin{bmatrix} 1 & .0625 \times 10^{-6} \\ -288\omega^2 & (1 - 18 \times 10^{-6} \omega^2) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{B1}^R$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_4^R = \begin{bmatrix} 1 & .01562 \times 10^{-6} \\ -96\omega^2 & (1 - 1.5 \times 10^{-6} \omega^2) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_3^R$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_2^R = \begin{bmatrix} 1 & .625 \times 10^{-6} \\ -10\omega^2 & (1 - 6.25 \times 10^{-6} \omega^2) \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{A1}^R$$

Solution

$$\omega_1 = 377.2 \quad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ -4.35 \end{Bmatrix}$$

$$\omega_2 = 427.0 \quad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ 2.605 \end{Bmatrix}$$

$$\omega_3 = 940.0 \quad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}^{(3)} = \begin{Bmatrix} 1.0 \\ 1.725 \end{Bmatrix}$$

12-40

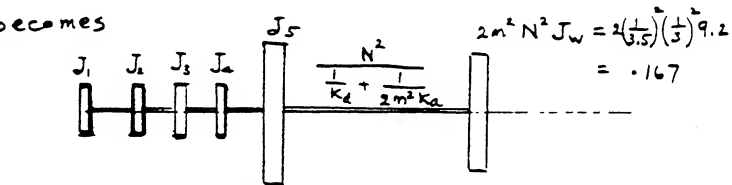
With $m = \frac{1}{3.5}$ The rear wheels and axle

are replaced by $2m^2 J_w$ and $2m^2 K_a$. In series with the drive shaft the stiffness is

$$\frac{1}{\frac{1}{K_d} + \frac{1}{2m^2 K_a}} \quad \text{Let } N = \frac{1}{3} = \text{transmission gear ratio,}$$

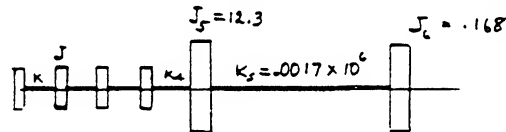
and refer all stiffnesses and inertias to engine speed by multiplying by N^2 . System then

becomes



$$\left. \begin{aligned} K_a &= \frac{(12 \times 10^6) \pi (1.25)^4}{32 \times 25} = .115 \times 10^6 \\ K_d &= \frac{(12 \times 10^6) \pi (1.5)^4}{32 \times 74} = .0806 \times 10^6 \end{aligned} \right\} K_s = \frac{(\frac{1}{3})^2 \times 10^6}{\frac{1}{.0806} + \frac{3.5}{.115}} = .00170 \times 10^6$$

12-41



$$J = J_1 = J_2 = J_3 = J_4 = 0.20 \text{ lb in sec}^2$$

$$K = 6.1 \times 10^6 \text{ lb in/rad}$$

$$K_d = 4.5 \times 10^6 \text{ lb in/rad}$$

Approximation to first nat. freq. as two mass system

$$\omega = \sqrt{\frac{K(J_1 + J_2)}{J_1 J_2}} = 10^3 \sqrt{\frac{.0017(13.1 + .168)}{13.1 \times .168}}$$

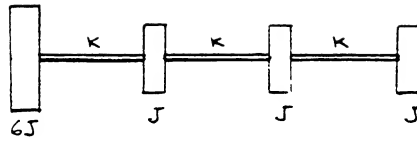
$$= 10^3 \sqrt{.0102} \approx 100 \text{ rad/sec}$$

computer solution follows

12-41 cont.

```
%This code is for problem 12-41
clear, clg;
m=2000;
cm=50;
step =20;
j=[.2 .2 .2 .2 12.3 .168];
k=1e6*[6.1 6.1 6.1 4.5 .0017];
for ii=1:m
    tl(ii,1)=0;
    theta(ii,1)=1;
    cmsq=cm^2;
    tr(ii,1)=-cmsq*j(1);
    for kk=1:5
        L=kk+1
        theta(ii,L)=theta(ii,kk)+tr(ii,kk)/k(kk);
        tl(ii,L)=tr(ii,kk);
        tr(ii,L)=tl(ii,L)-cmsq*j(L)*theta(ii,L);
    end
    cm=cm+step;
end
```

12-42



The system is first reduced to the above model

The equations in matrix form are

$$K \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = -\omega^2 J \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

Rearrange to

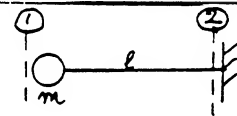
$$\frac{K}{\omega^2 J} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1.667 & .1667 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix}$$

Results of iteration with sweeping matrix

$$\begin{array}{c|c} \omega_0 = 0, \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix} & \omega_2 = 1.272 \sqrt{\frac{K}{J}}, \begin{Bmatrix} 1.00 \\ -1.614 \\ -1.237 \\ .1416 \end{Bmatrix} \\ \hline \omega_1 = .5375 \sqrt{\frac{K}{J}}, \begin{Bmatrix} 1.00 \\ .714 \\ .239 \\ -.326 \end{Bmatrix} & \omega_3 = 1.825 \sqrt{\frac{K}{J}}, \begin{Bmatrix} 1.00 \\ -2.27 \\ 1.87 \\ -.1005 \end{Bmatrix} \end{array}$$

12-43

From Eq. 12.9-5



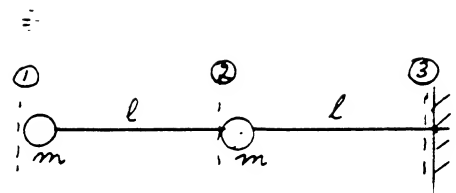
$$\begin{Bmatrix} -V \\ M \\ 0 \\ 0 \end{Bmatrix}_2 = \begin{bmatrix} - & - & - & - \\ & & & 1 - \frac{m\omega^2 l^2}{2EI} \\ & & l & (1 + \frac{m\omega^2 l^3}{6EI}) \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ 1 \end{Bmatrix}_1$$

$$\therefore \begin{vmatrix} 1 & \frac{m\omega^2 l^2}{2EI} \\ l & (1 + \frac{m\omega^2 l^3}{6EI}) \end{vmatrix} = 0 \quad \omega = \sqrt{\frac{3EI}{ml^3}}$$

12-44

From Eq. 12.9-5

Let $\alpha = l/6EI$



$$\begin{Bmatrix} -V \\ M \\ 0 \\ 0 \end{Bmatrix}_3 = \begin{bmatrix} 1 & 0 & 0 & m\omega^2 \\ l & 1 & 0 & m\omega^2 l \\ 3\alpha l & 6\alpha & 1 & 3m\omega^2 \alpha l \\ \alpha l^2 & 3\alpha l & l & (1+m\omega^2 \alpha l^2) \end{bmatrix}^2 \begin{Bmatrix} 0 \\ 0 \\ \theta \\ 1 \end{Bmatrix}_1$$

We need only to calculate the last two columns of the last two rows, which are:

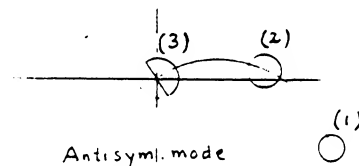
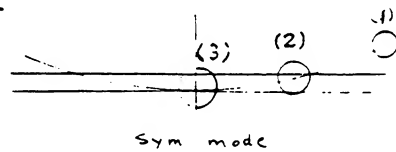
$$\begin{Bmatrix} (1+3m\omega^2 \alpha l^2) & [15m\omega^2 \alpha l + 3(m\omega^2 \alpha)^2 l^3] \\ (2l + m\omega^2 \alpha l^3) & [1+9m\omega^2 \alpha l^2 + (m\omega^2 \alpha l^2)^2] \end{Bmatrix} \begin{Bmatrix} \theta \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Det. set to zero gives

$$(m\omega^2 \alpha l^2)^2 - 3(m\omega^2 \alpha l^2) + \frac{1}{6} = 0$$

Solving $\omega = \begin{cases} 0.583 \sqrt{\frac{EI}{m l^3}} \\ 4.20 \sqrt{\frac{EI}{m l^3}} \end{cases}$

12-45



$$\begin{Bmatrix} 0 \\ M \\ 0 \\ y \end{Bmatrix}_3 = \begin{bmatrix} u_{ij} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ y \end{Bmatrix}_1$$

Freq. eq.

$$\begin{vmatrix} u_{13} & u_{14} \\ u_{33} & u_{34} \end{vmatrix} = 0$$

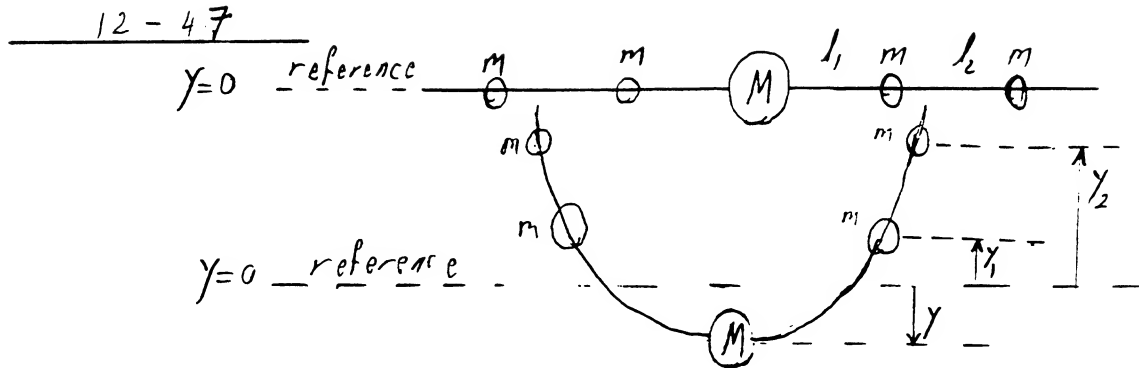
$$\begin{Bmatrix} -V \\ 0 \\ \theta \\ 0 \end{Bmatrix}_3 = \begin{bmatrix} u_{ij} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \theta \\ y \end{Bmatrix}_1$$

Freq. eq.

$$\begin{vmatrix} u_{23} & u_{24} \\ u_{43} & u_{44} \end{vmatrix} = 0$$

12-46

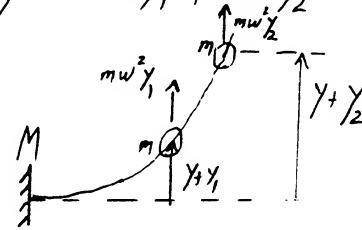
The solution is the same as problem 12-20.



Conservation of momentum gives

$$M\dot{y} = 2m\dot{y}_1 + 2m\dot{y}_2 \xRightarrow{\text{integration}} (1) \quad My = 2m y_1 + 2m y_2$$

$$y(x) = \begin{cases} \frac{Px^2}{6EI} (3a-x), & 0 \leq x \leq a \\ \frac{Pa^2}{6EI} (3x-a), & a \leq x \leq l \end{cases}$$

 \therefore Using the superposition principle

$$(2) \quad y + y_1 = \frac{(m\omega^2 y_1) l_1^3}{3EI} + \frac{(m\omega^2 y_2) l_1^2 (3l_2 - l_1)}{6EI}$$

$$(3) \quad y + y_2 = \frac{(m\omega^2 y_1) l_1^2 (3l_2 - l_1)}{6EI} + \frac{(m\omega^2 y_2) l_2^3}{3EI}$$

From (1) and (2) solve for y and y_1 in terms of y_2

$$y = \frac{2}{n} y_1 + \frac{2}{n} y_2$$

$$(2) \text{ becomes } \left(1 + \frac{2}{n}\right) y_1 + \frac{2}{n} y_2 = \frac{(m\omega^2 y_1) l_1^3}{3EI} + \frac{(m\omega^2 y_2) l_1^2 (3l_2 - l_1)}{6EI}$$

$$\therefore y_1 = \frac{1}{1 + \frac{2}{n} - \frac{m\omega^2 l_1^3}{3EI}} \cdot \left[\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n} \right] y_2$$

12-47 cont.

$$Y = \left[\frac{2/n}{1 + \frac{2}{n} - \frac{m\omega^2 l_1^3}{3EI}} \cdot \left[\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n} \right] + \frac{2}{n} \right] \frac{Y}{2}$$

From (3), we get

$$\left[\frac{2/n}{1 + \frac{2}{n} - \frac{m\omega^2 l_1^3}{3EI}} \cdot \left[\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n} \right] + \frac{2}{n} + 1 \right] =$$

$$\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} \cdot \frac{1}{1 + \frac{2}{n} - \frac{m\omega^2 l_1^3}{3EI}} \cdot \left[\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n} \right] + \frac{m\omega^2 l_2^3}{3EI}$$

$$\therefore \frac{2}{n} \left[\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n} \right] + \frac{2}{n} + 1 = \frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} \left[\frac{m\omega^2 l_1^2 (3l_2 - l_1)}{6EI} - \frac{2}{n} \right] +$$

$$\left(1 + \frac{2}{n} - \frac{m\omega^2 l_1^3}{3EI} \right) \frac{m\omega^2 l_2^3}{3EI}$$

$$\Rightarrow A\omega^4 + B\omega^2 + C = 0, \text{ where,}$$

$$A = \left[\frac{m l_1^2 (3l_2 - l_1)}{6EI} \right]^2 - \frac{m^2 l_1^3 l_2^3}{(3EI)^2}$$

$$B = \left(1 + \frac{2}{n} \right) \frac{m l_2^3}{3EI} - \frac{2}{n} \frac{m l_1^2 (3l_2 - l_1)}{3EI}$$

$$C = \frac{4}{n^2} - \frac{2}{n} - 1$$

$$\therefore \omega_{1,2}^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\omega_{1,2} = \sqrt{\omega_{1,2}^2} \quad \text{with } \omega_{1,2}^2 \geq 0. \text{ Ignore } \omega_{1,2}^2 < 0$$

13-1

- (1) noise in radio reception - wide freq. range & probably stationary.
- (2) ground motion of earthquake - non stationary
- (3) ocean wave heights - non stationary and dependent on wind or sea state

13-2

See text section 13.2.

13-3

See text sections 13.2-13.3 for definition $= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i$

$E(L)$ for 800 coins thrown 100 times = 400

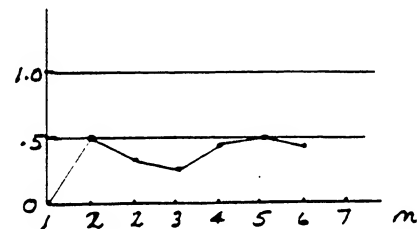
" " 8 " " " = 4

" " 8 " " 1000 " = 4

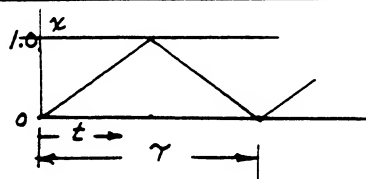
13-4

Make table as suggested here

Throw	H	T	sum	sum ÷ n
n = 1		0	0	0
2	1		1	1/2
3		0	1	1/3
4		0	1	1/4
5	1		2	2/5
6	1		3	3/6
7		0	3	3/7
etc				



13-5



$$x = \frac{2t}{T} \quad t \leq T/2$$

$$x = 2 - \frac{2t}{T} \quad t \geq T/2$$

Mean value = 0.50

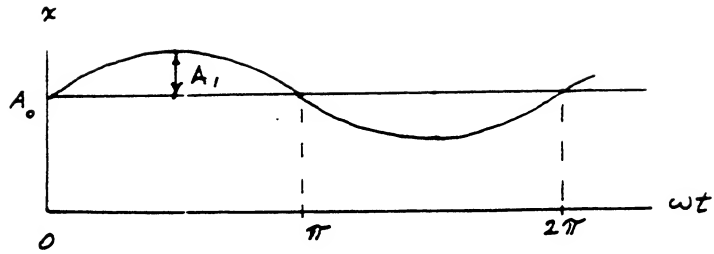
Mean square value

$$\overline{x^2} = \frac{1}{T} \left\{ \int_0^{T/2} \left(\frac{2t}{T} \right)^2 dt + \int_{T/2}^T \left[2 - \frac{2t}{T} \right]^2 dt \right\}$$

$$= \frac{1}{T} \left[\frac{4}{T^2} \cdot \frac{t^3}{3} \right]_0^{T/2} + \left(4t - \frac{8}{T} \frac{t^2}{2} + \frac{4}{T^2} \frac{t^3}{3} \right) \Big|_{T/2}^T$$

$$= \frac{1}{3}$$

13-6



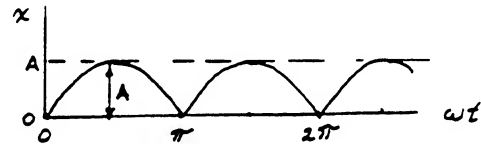
$x = A_0 + A_1 \sin wt$ - Mean val. = A_0 by inspection

Let $wt = \theta$

$$\begin{aligned}\bar{x} &= \frac{1}{2\pi} \int_0^{2\pi} (A_0^2 + 2A_0 A_1 \sin \theta + A_1^2 \sin^2 \theta) d\theta \\ &= \frac{1}{2\pi} \left\{ A_0^2 \theta - 2A_0 A_1 \cos \theta + \frac{A_1^2}{2} \theta - \frac{A_1^2}{2} \frac{\sin 2\theta}{2} \right\}_0^{2\pi} \\ &= \frac{1}{2\pi} \left\{ A_0^2 2\pi - 2A_0 A_1 (1-1) + \frac{A_1^2}{2} 2\pi - 0 \right\} = A_0^2 + \frac{1}{2} A_1^2\end{aligned}$$

13-7

$$x = |A \sin wt|$$



$$\bar{x} = \frac{A}{\pi} \int_0^{\pi} \sin \theta d\theta = \frac{A}{\pi} (-\cos \theta) \Big|_0^{\pi} = \frac{2A}{\pi}$$

$$\bar{x}^2 = \frac{A^2}{\pi} \int_0^{\pi} \sin^2 \theta d\theta = \frac{A^2}{\pi} \frac{\theta}{2} \Big|_0^{\pi} = \frac{A^2}{2}$$

13-8

Peak values are positive quantities \therefore cannot have a probability in the negative region. Also probability of zero or infinite peaks is zero.

13-9

Area under normal probability curve = 1.0

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma \sqrt{2\pi}} dx = \frac{\sigma \sqrt{2\pi}}{\sigma \sqrt{2\pi}} = 1.0 \quad (a)$$

13-9 Cont.

For higher moments let $\alpha = \frac{1}{2\sigma^2}$ and

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \quad \dots \dots \dots (b)$$

From (a) $I = \sigma \sqrt{2\pi} = \sqrt{\pi} \alpha^{-\frac{1}{2}}$

Start with $I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi} \alpha^{-\frac{1}{2}} \quad (c)$

Differentiate w.r.t. α

$$\begin{aligned} \frac{\partial I}{\partial \alpha} &= - \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{1}{2} \sqrt{\pi} \alpha^{-\frac{3}{2}} \quad (d) \\ &= -\frac{1}{2} \sqrt{\pi} (2\sigma^2)^{\frac{3}{2}} = -\sqrt{2\pi} \sigma^3 \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 e^{-\alpha x^2}}{\sigma \sqrt{2\pi}} dx = \underline{\underline{\sigma^2 = E(x^2)}}$$

Differentiate w.r.t. α again starting with (d)

$$\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = -\frac{1}{2} \sqrt{\pi} \left\{ -\frac{3}{2} \alpha^{-\frac{5}{2}} \right\} = \frac{3}{4} \sqrt{\pi} (2\sigma^2)^{\frac{5}{2}}$$

$$\therefore \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \underline{\underline{E(x^4) = 3\sigma^4}}$$

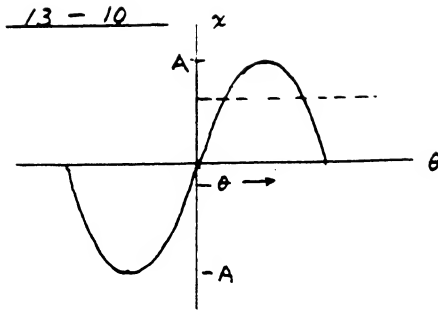
Repeat

$$\begin{aligned} \frac{\partial}{\partial \alpha} \left\{ \frac{3}{4} \sqrt{\pi} \alpha^{-\frac{5}{2}} \right\} &= -\frac{3}{4} \cdot \frac{5}{2} \sqrt{\pi} \alpha^{-\frac{7}{2}} = -\frac{3}{4} \cdot \frac{5}{2} \sqrt{\pi} (2\sigma^2)^{\frac{7}{2}} \\ &= -3 \cdot 5 \sqrt{2\pi} \sigma^7 \end{aligned}$$

$$\div \text{ by } \sigma \sqrt{2\pi} \quad \underline{\underline{E(x^6) = 3 \cdot 5 \sigma^6}}$$

General Eq. is $\underline{\underline{E(x^n) = 1 \cdot 3 \cdot 5 \cdot \dots (n-1) \sigma^n}} \quad (\text{for } n \text{ even})$

by inspection $E(x^n) = 0$ for n odd.



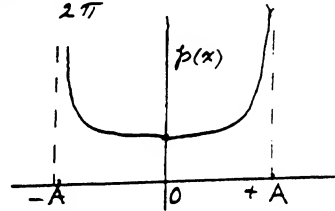
$$x = A \sin \theta$$

When $x=0$ $P(x) = \frac{1}{2}$
i.e. half the time x is less than $x=0$.

As we increase x from $x=0$ we add $\frac{2\theta}{2\pi}$ to the probability,

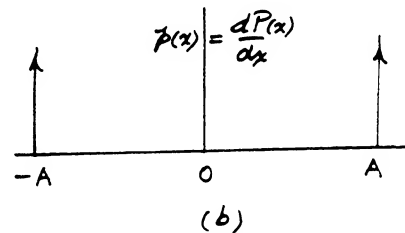
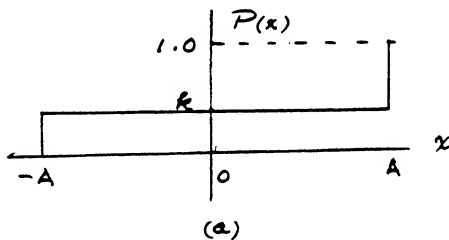
$$\theta = \sin^{-1} \frac{x}{A}$$

$$P(x) = \frac{1}{2} \pm \sin^{-1} \frac{x}{A}$$



$$p(x) = \frac{dP(x)}{dx} = \frac{1}{\pi} \frac{d}{dx} \left(\sin^{-1} \frac{x}{A} \right) = \frac{1}{\pi \sqrt{A^2 - x^2}}$$

13-11 Measure the total length of line at $-A$ and divide by total length at A and $-A$. Let this fraction be k ; then the cumulative prob. curve will appear as in (a). Density curve is shown in (b)



13-12

$$x(t) = A \cos t$$

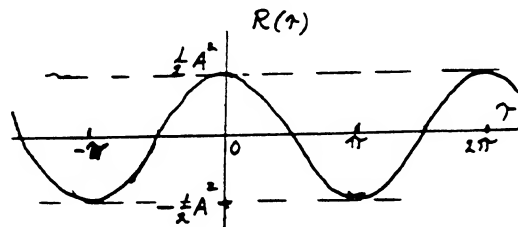
$$x(t+\tau) = A \cos(t+\tau) = A[\cos t \cos \tau - \sin t \sin \tau]$$

$$x(t)x(t+\tau) = A^2 [\cos^2 t \cos \tau - \cos t \sin t \sin \tau]$$

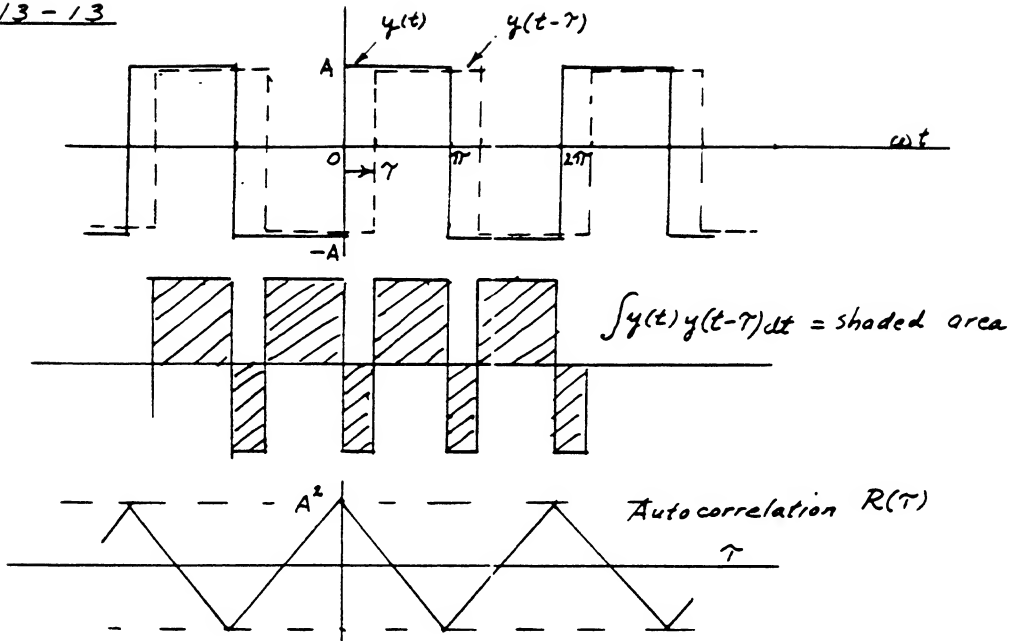
$$R(\tau) = \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\pi/2}^{\pi/2} [\cos \tau \cdot \frac{1}{2}(1 + \cos 2t) - \sin \tau \cdot \sin t \cos t] dt$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{T} \left[\cos \tau \cdot \left(\frac{\pi}{2} \right) - \sin \tau \cdot (0) \right]$$

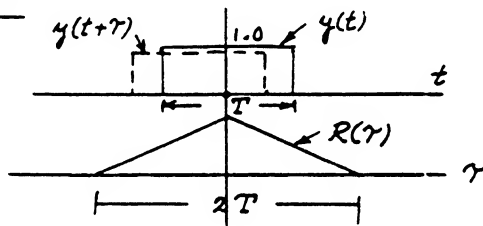
$$= \frac{A^2}{2} \cos \tau$$



13-13

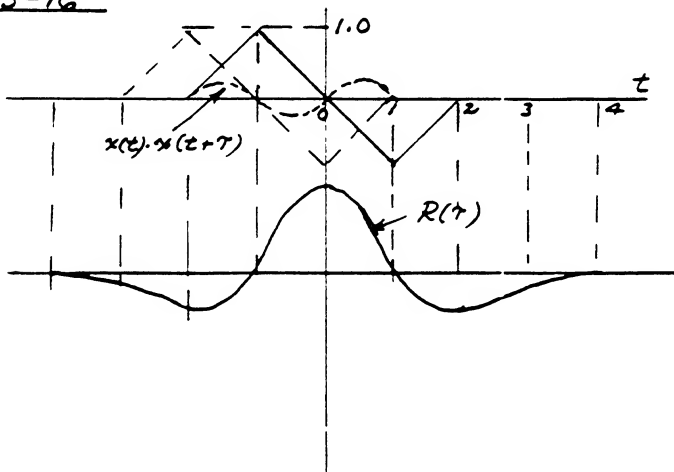


13-14



13-15 This problem is similar to Prob. 13-14. Integrate the area under $y(t) \cdot y(t+\tau)$ curve. Start curve with $R(\tau) = 5$ at $\tau = 0$ and linearly decrease to $R(\tau) = 1.0$ at $\tau = 1$, etc. Shift traced curve as suggested.

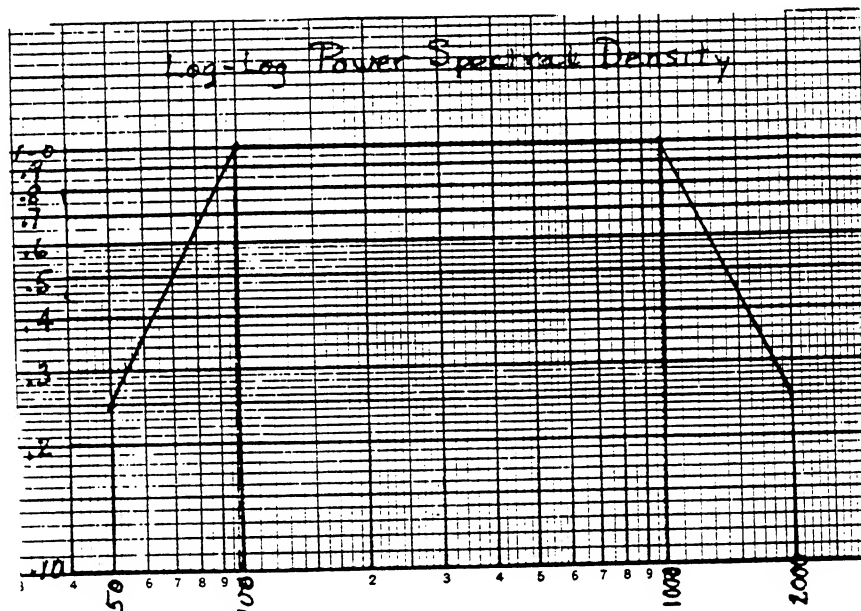
13-16



$$\begin{aligned}
 \underline{13-17} \quad \bar{x}^2 &= .20 \frac{g^2}{Hz} \times 500 Hz = 100 g^2 \\
 &= 100 \times 9.81^2 = 9623.6 \\
 RMS = \bar{x} &= \sqrt{9623.6} = 98.1 \text{ m/s}^2
 \end{aligned}$$

$$\begin{aligned}
 \underline{13-18} \quad Area &= 1 \times 900 + 2 \times 1000 = 2900 g^2 \\
 RMS &= 9.81 \sqrt{2900} = 528.3 \text{ m/s}^2
 \end{aligned}$$

13-19

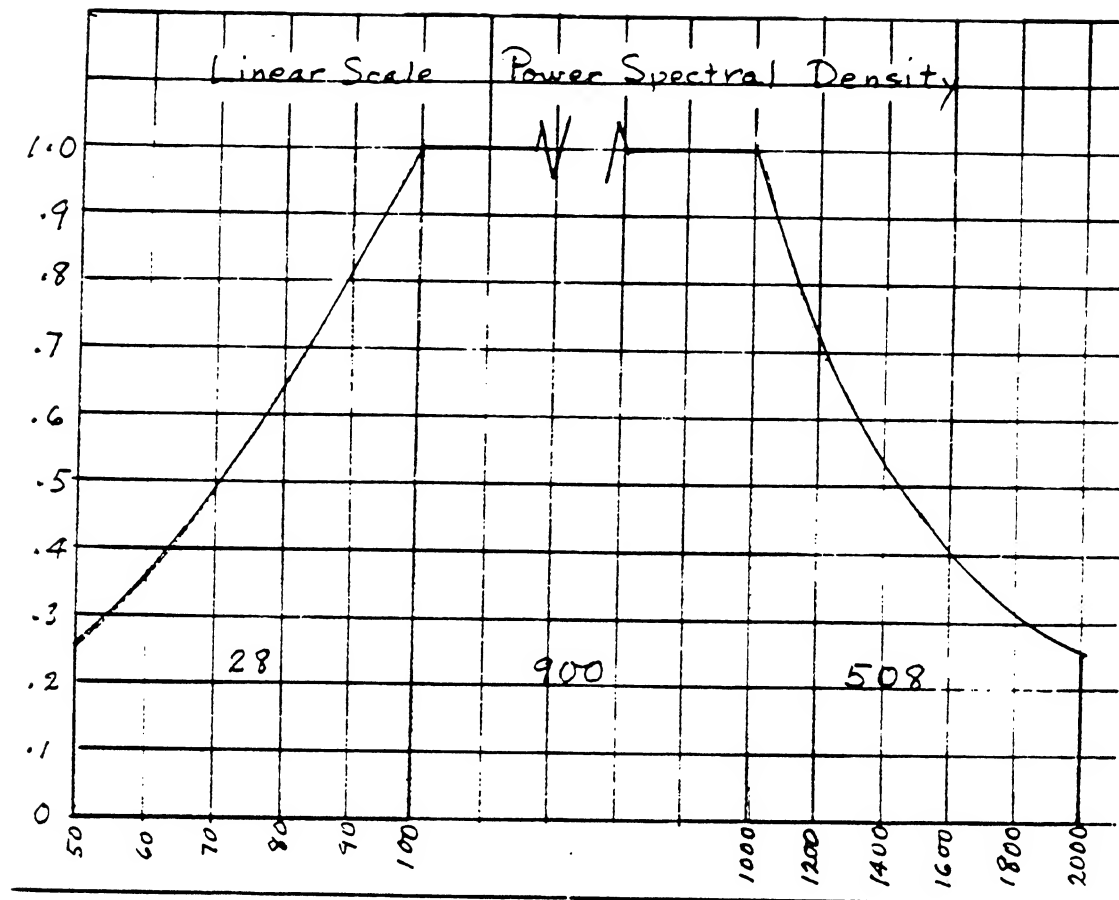


$$DB = 10 \log_{10} \frac{1.0}{0.25} = 6.02 \text{ db/octave}$$

The log-log plot is replotted on linear scale

$$\bar{x}^2 = \text{total area} = \underline{\underline{1436}} \quad RMS = 37.9 \text{ m/s}^2$$

13-19 Cont:



13-20 Procedure here is to determine the Fourier series and plot the quantity $\frac{1}{2} C_n C_n^*$ where $C_n = a_n - i b_n$. Note that $C_n C_n^* = a_n^2 + b_n^2$ i.e. for Fig. 13-20 (a), $x(t) = \frac{1}{2} - \frac{1}{\pi} [\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \dots]$ (see prob. 1-12.) $\therefore a_n = 0, b_n = \frac{1}{n\pi}$

$$\sum \frac{1}{2} C_n C_n^* = \frac{1}{4} + \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \dots$$



13-21

$$\overline{x^2} = .002 \times 1980 = 3.96 \text{ in}^2$$

$$\text{RMS} = \sqrt{\overline{x^2}} = 1.99 \text{ in}$$

$$\sigma^2 = \overline{x^2} - (\bar{x})^2$$

$$= 3.96 - (1.732)^2 = 0.9602$$

$$\sigma = 0.9799$$



13-22

C_n found by multiplying $f(t)$ by $e^{-in\omega_0 t}$ and integrating over one period

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

13-23

$$C_n = \frac{1}{2} (a_n - i b_n)$$

$$C_0 = \frac{1}{2} a_0$$

$$\text{Re}(C_n e^{in\omega_0 t}) = \frac{1}{2} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t})$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (C_n e^{in\omega_0 t} + C_n^* e^{-in\omega_0 t})$$

but $C_n^* = C_{-n} \therefore$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2} C_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

$$C_n = 2C_n \text{ (see Eq. 13.2-9)}$$

13-24

Same procedure as Prob 13-20

ie Fig P13-24 (See Prob 1-11)

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2} (\cos \omega_1 t + \frac{1}{3^2} \cos 3\omega_1 t + \dots)$$

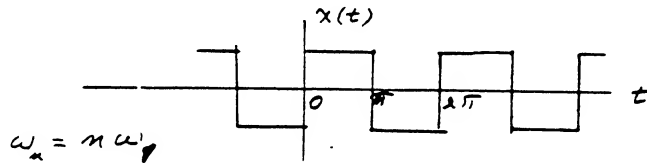
$$S(f) = \sum \frac{C_n C_n^*}{2} = \sum \bar{C}_n^2$$

13-25

See Ch. 1 Sec. 1.2

13-26

$$x = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}$$



$$\omega_n = n\omega_1$$

$$C_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t) e^{-i\omega_n t} dt = \frac{1}{2} (a_n - i b_n), \quad C_0 = \frac{A_0}{2}$$

Let $\omega_n t = n\omega_1 t = n\theta$ where $\begin{cases} \theta = \omega_1 t \\ d\theta = \omega_1 dt \end{cases}$

Limits $n\omega_1 \frac{\tau}{2} = n\theta$

$n \frac{2\pi}{\tau} \frac{\tau}{2} = n\pi = n\theta \quad \therefore \theta \text{ goes betw. } \pm \pi$

$$C_n = \frac{1}{\tau} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \frac{d\theta}{\omega_1}$$

$$\omega_1 \tau = \frac{2\pi}{\tau} \cdot \tau = 2\pi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta = \frac{2A}{i n \pi} \text{ for } n \text{ odd}$$

$$= 0 \text{ for } n \text{ even}$$

$$\therefore x(t) = \frac{2A}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in\omega_1 t}}{in} = \frac{4A}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{e^{in\omega_1 t} - e^{-in\omega_1 t}}{2i} \right)$$

$$= \frac{4A}{\pi} \left[\sin \omega_1 t + \frac{1}{3} \sin 3\omega_1 t + \frac{1}{5} \sin 5\omega_1 t + \dots \right]$$

plot $\frac{1}{2}$ (square of ampl.)

$$\frac{13-27}{\left(\frac{xk}{F}\right)} = \frac{1}{\sqrt{(1-\eta^2)^2 + (25\eta)^2}} \quad \text{where } \eta = \frac{f}{f_m} = \frac{\omega}{\omega_m}$$

At $\eta = 1$ $\frac{xk}{F} = \frac{1}{25}$ At half power pt. $\frac{xk}{F} = \frac{1}{\sqrt{2}} \cdot \frac{1}{25}$

$$\therefore \frac{1}{2} \left(\frac{1}{25}\right)^2 = \frac{1}{(1-\eta^2)^2 + (25\eta)^2}$$

$$\eta^4 - 2\eta^2 + 1 + 45^2 \eta^2 = 85^2$$

$$\eta^4 - 2(1-25^2)\eta^2 + (1-85^2) = 0$$

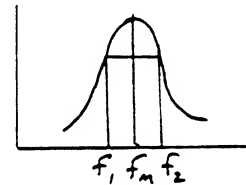
$$\therefore \eta^2 = (1-25^2) \pm 25\sqrt{1+5^2}$$

$$\therefore \eta \approx (1 \pm 25)^{1/2} = 1 \pm 5 + \text{negl. term}$$

$$\left\{ \begin{array}{l} \text{for } 5 \ll 1 \\ \eta^2 \approx 1 \pm 25 \\ \text{by neglecting } 5^2 \end{array} \right.$$

$$\therefore f_1 = f_m(1-5) = f_m\left(1 - \frac{1}{2Q}\right)$$

$$f_2 = f_m(1+5) = f_m\left(1 + \frac{1}{2Q}\right)$$



13-28

$$\int_0^{\infty} \frac{d\eta}{(1-\eta^2)^2 + (25\eta)^2} \quad \text{where } \eta = \frac{f}{f_m}$$

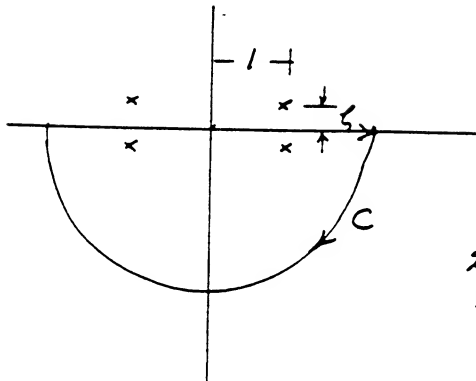
1st find poles from zeros of denominator

$$F(\eta) = \eta^4 - 2(1-25^2)\eta^2 + 1 = 0$$

$$\eta^2 = (1-25^2) \pm i 25\sqrt{1-5^2} \approx 1 \pm i 25 \quad \text{for } 5 \ll 1$$

See Theory of Residues p 135-137 Laplace Transformation
by W.T. Thomson - Prentice-Hall, Inc.

13-28 Cont:



Form contour with infinite circle. Poles are $\eta \approx 1 \pm i5$

Integrate around contour

C

$$2 \int_0^{\infty} \dots \int_C = 2\pi i \sum \text{Residues within contour}$$

Residues at the two poles are

$$\frac{1}{F'(\eta)} = \frac{1}{2(1-\eta^2)(-2\eta) + 85^2 \eta} \quad \text{evaluated at } \eta_1 = 1 - i5 = -(1 + i5)$$

$$\approx i \frac{1}{45} \quad \text{and } \eta_1^2 = 1 - i25$$

$$\eta_2^2 = 1 + i25$$

Since $\int_C = 0$ we have

$$-2 \int_0^{\infty} = 2\pi i \left(\frac{i}{45} \right) = -\frac{\pi}{25}$$

$$\therefore \int_0^{\infty} \frac{d\eta}{(1-\eta^2)^2 + (25\eta)^2} = \frac{\pi}{45}$$

Can also be checked
by numerical integration
(must use very small $\Delta\eta$)

13-29

from definition

$$\bar{X}(s) = \frac{F(s)}{ms^2 + k(1+i\gamma)} = \bar{H}(s) \bar{F}(s)$$

13-30

$$H(\omega) = \frac{1}{k} \frac{1}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + [2\zeta \frac{\omega}{\omega_n}]^2}}$$

Each component can be treated separately. For 1st component $F \cos(.5\omega_n t - \theta_1)$, the mean square response is

$$\frac{1}{[1 - (.5)^2]^2 + [.2 \times .5]^2} \times \frac{1}{2} \left(\frac{F}{k}\right)^2 = 1.746 \times \frac{1}{2} \left(\frac{F}{k}\right)^2$$

Similarly other components are

$$2^{\text{nd}} \text{ comp } \frac{1}{[.2]^2} \times \frac{1}{2} \left(\frac{F}{k}\right)^2 = 25 \times \frac{1}{2} \left(\frac{F}{k}\right)^2$$

$$3^{\text{rd}} \text{ comp } \frac{1}{[1 - 2^2]^2 + [.2 \times 2]^2} \times \frac{1}{2} \left(\frac{F}{k}\right)^2 = .109 \times \frac{1}{2} \left(\frac{F}{k}\right)^2$$

$$\therefore \bar{x}^2 = [1.746 + 25 + .109] \times \frac{1}{2} \left(\frac{F}{k}\right)^2 = 26.85 \times \frac{1}{2} \left(\frac{F}{k}\right)^2 \\ = 13.43 \left(\frac{F}{k}\right)^2$$

13-31

From Sec 13.4 & Ex. 13.8-1, $\sigma = 26.6 \text{ g}$
 $2\sigma = 53.2 \text{ g}$

$$P[-2\sigma \leq x(t) \leq 2\sigma] = 95.4\%$$

$$\therefore P[x(t) \geq 2\sigma] = 100 - 95.4 = \underline{\underline{4.6\%}}$$

$$P[\bar{X} \geq 2\sigma] = \underline{\underline{13.5\%}}$$

13-32

$$m = 40 \text{ kg.}$$

$$\omega_n = 2\pi \times 2.20 = 13.82$$

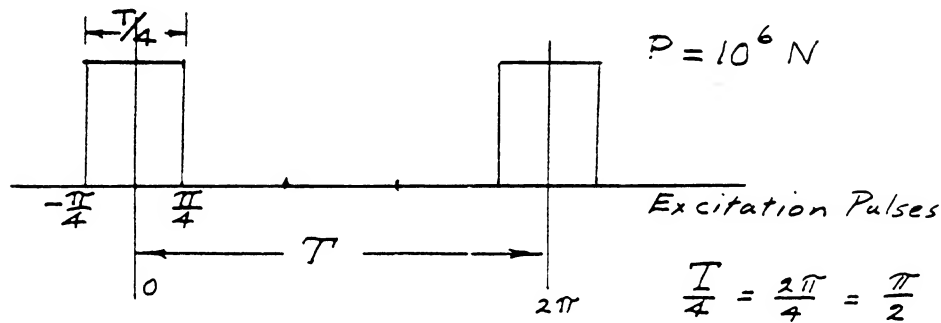
$$\text{Assume } \zeta = .15$$

$$k = m\omega_n^2 = 40 \times 13.82^2 = 7643.$$

$$f_n = 2.20 \text{ Hz}$$

$$\omega_1 = \frac{2\pi}{T}$$

$$X = \frac{F_0}{k} \frac{\sin(\omega t - \phi)}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2]^2 + [2\zeta \frac{\omega}{\omega_n}]^2}}$$



F.S. of the excitation is a cosine series

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_1 t dt = \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} P \cos n\theta \frac{d\theta}{\omega_1}$$

$$= \frac{P}{\pi\omega_1} \frac{\sin n\theta}{n} \bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{P}{n\pi\omega_1} \right) 2 \sin \frac{n\pi}{4}$$

$$= \left(\frac{2P}{\pi\omega_1} \right) \left(\frac{\sin \frac{n\pi}{4}}{n} \right)$$

$$a_0 = \frac{2}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} P dt = \frac{P}{2}$$

n	$\frac{\sin \frac{n\pi}{4}}{n}$
1	$\sqrt{2}/2$
2	$1/2$
3	$\sqrt{2}/6$
4	0
5	$-\sqrt{2}/10$
6	$-1/6$

13-32 Cont.

Excitation Spectrum $S_p(\omega)$

$$S_p(\omega) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \frac{a_n^2}{2} = \frac{P^2}{8} + \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{2P}{\pi \omega_n} \right)^2 \frac{\sin^2 \frac{n\pi}{4}}{n^2}$$

$$= 10^{12} \left[\frac{1}{8} + \left(\frac{2}{\pi^2 \omega_n^2} \right) \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 0 + \frac{1}{50} + \dots \right) \right]$$

Response Spectrum $\bar{y}^2 = \int_0^{\infty} H H^* S_p df = \bar{x}^2$

$$= \sum H H^* S_p$$

$$H H^* = \frac{1}{k^2} \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[2\zeta \frac{\omega}{\omega_n} \right]^2}$$

$$\omega = n\omega_n = n \frac{2\pi}{T} \quad \omega_n = 2\pi f_n = 2\pi \cdot 2.2$$

$$\frac{\omega}{\omega_n} = n \left(\frac{1}{2.2T} \right) \quad \text{subst. into } \bar{x}^2$$

13-33 For base motion, the relative motion is given by Eq. 3.5-4, which can be written as

$$Z = \frac{\left(\frac{\omega}{\omega_n} \right)^2 Y}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[2\zeta \frac{\omega}{\omega_n} \right]^2}}$$

Thus with $S(f) = \text{Spectral density of excitation}$

$$\bar{Z}^2 = \int_0^{\infty} S_Y(f_r) \frac{\left(\frac{f}{f_n} \right)^4 df}{\left[1 - \left(\frac{f}{f_n} \right)^2 \right]^2 + \left[2\zeta \frac{f}{f_n} \right]^2} = S_Y(f_n) \cdot f_n \cdot \int_0^{\infty} \frac{\xi^2 d\xi}{\left[1 - \xi^2 \right]^2 + \left[2\zeta \xi \right]^2}$$

\therefore must evaluate $\int_0^{\infty} \frac{\xi^2 d\xi}{\left[1 - \xi^2 \right]^2 + \left[2\zeta \xi \right]^2}$

13-34

$$\ddot{x} = \frac{(k + i\omega c)}{k - m\omega^2 + i\omega c} \ddot{y} = \frac{1 + i(25 \frac{f}{f_n})}{1 - (\frac{f}{f_n})^2 + i(25 \frac{f}{f_n})} \ddot{y}$$

$$\begin{aligned} \overline{\ddot{x}^2} &= \int_0^\infty f_n \overline{\ddot{y}^2} \frac{1 + i(25 \frac{f}{f_n})}{1 - (\frac{f}{f_n})^2 + i(25 \frac{f}{f_n})} \cdot \frac{1 - i(25 \frac{f}{f_n})}{1 - (\frac{f}{f_n})^2 - i(25 \frac{f}{f_n})} \frac{df}{f_n} \\ &= \int_0^\infty S_{\ddot{y}}(f) \cdot f_n \cdot \frac{1 + (25 \frac{f}{f_n})^2}{[1 - (\frac{f}{f_n})^2]^2 + [25 \frac{f}{f_n}]^2} \cdot d(\frac{f}{f_n}) \end{aligned}$$

13-35

$$\omega^2 = \frac{k}{m} = \frac{k}{60} = (2\pi \cdot 4)^2$$

$$k = 60 \times (8\pi)^2 = 37,899 \text{ N/m}$$

$$k^2 = 1436 \times 10^6 \text{ N}^2/\text{m}^2$$

$$H^2 = \frac{1}{1436 \times 10^6 \left\{ \left[1 - \left(\frac{\omega}{8\pi} \right)^2 \right]^2 + \left[2(0.05) \frac{\omega}{8\pi} \right]^2 \right\}}$$

$$\overline{y^2} = \int_0^\infty H^2 S(\omega) d\omega \approx \frac{S(\omega_n)}{k^2} f_n \frac{\pi}{45}$$

$$= \frac{100 \times 10^3}{1436 \times 10^6} \cdot \frac{\pi}{45} = .00438$$

$$\sigma^2 = \overline{y^2} = .00438 \text{ m}^2, \quad \sigma = .0662 \text{ m}$$

$$y = .132 = 1.99 \sigma$$

$$P[y > 1.99\sigma] = \underline{\underline{4.6\%}}$$

13-36

$$m = 272 \text{ kg}$$

$$f_m = 26 \text{ Hz}$$

$$\omega_m = 2\pi f_m = 163,36$$

$$\zeta = 0,10$$

$$k = m\omega^2 = 272 \times (163,36)^2$$

$$= 7258 \times 10^3 \text{ N/m}$$

$$k^2 = 52.689 \times 10^{12}$$

$$\overline{x^2} = \sigma^2 = \frac{S(f_m)}{k^2} \cdot f_m \cdot \frac{\pi}{4\zeta}$$

$$\sigma^2 = \frac{4 \times 10^6}{52.689 \times 10^{12}} \times 26 \times \frac{\pi}{4 \times 0,10} = 15,50 \times 10^{-6}$$

$$\sigma = 0,003937$$

$$0,012 \text{ m} = 3,05 \sigma$$

$$P[|x| > 3,05\sigma] = 0,3\%$$

13-37

$$S(f) = 5 \times 10^6 \text{ N}^2/\text{Hz}, \quad \zeta = 0,03$$

$$\omega_m = 30$$

$$f_m = \frac{\omega_m}{2\pi} = 4,775 \text{ Hz}$$

$$m = 1500 \text{ kg}, \quad k = m\omega^2 = 1500 \times 30^2 = 1,350 \times 10^6$$

$$k^2 = 1,823 \times 10^{12}$$

$$\sigma^2 = \overline{x^2} = \frac{S(f_m)}{k^2} \cdot f_m \cdot \frac{\pi}{4\zeta} = \frac{5 \times 10^6}{1,823 \times 10^{12}} \times 4,775 \times \frac{\pi}{4 \times 0,03}$$

$$= 342,9 \times 10^{-6}$$

$$\sigma = 0,01852$$

$$0,037 = 2\sigma$$

$$P[A > 0,037] = P[A > 2\sigma] = 13,5\%$$

13-38

$$\begin{aligned}x(t) &= \int_0^{\infty} f(t-\xi) g(\xi) d\xi \\X(i\omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \int_0^{\infty} f(t-\xi) g(\xi) d\xi e^{-i\omega t} dt \\&= \int_{-\infty}^{\infty} \int_0^{\infty} f(t-\xi) e^{-i\omega(t-\xi)} dt g(\xi) e^{-i\omega\xi} d\xi \\X(i\omega) &= \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(t-\xi) e^{-i\omega(t-\xi)} dt \right] g(\xi) e^{-i\omega\xi} d\xi \\&\quad \text{Let } (t-\xi) = \tau \quad dt = d\tau \\&= \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau \right] g(\xi) e^{-i\omega\xi} d\xi \\&= \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau \int_0^{\infty} g(\xi) e^{-i\omega\xi} d\xi = F(i\omega) H(i\omega) \\\overline{x^2} &= \int_0^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi T} X(i\omega) X^*(i\omega) d\omega \\&= \int_0^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi T} F(i\omega) F^*(i\omega) H(i\omega) H^*(i\omega) d\omega \\&= \int_0^{\infty} S_F(\omega) |H(i\omega)|^2 d\omega\end{aligned}$$

13-39

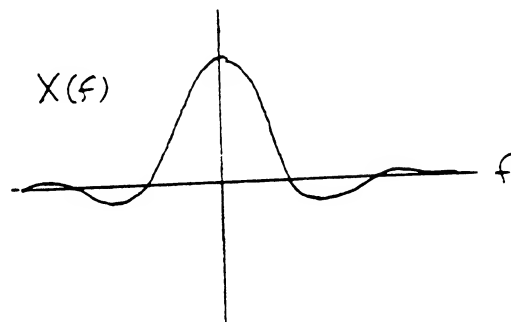
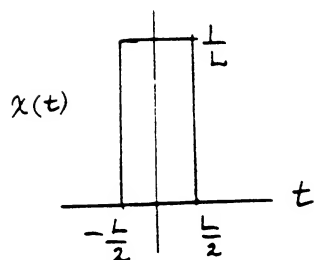
$$\begin{aligned}H(i\omega) &= |H(i\omega)| e^{i\phi(\omega)} \\H^*(i\omega) &= |H(i\omega)| e^{-i\phi(\omega)} \\\therefore \frac{H(i\omega)}{H^*(i\omega)} &= e^{i2\phi(\omega)}\end{aligned}$$

13-40

Let $X(f) = \text{F.T. of rectangular pulse}$

$$X(f) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{L}\right) e^{-2\pi f t} dt = \left(\frac{1}{L}\right) \frac{e^{-i2\pi f t}}{-i2\pi f} \Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{1}{L} \frac{1}{\pi f} \left[\frac{e^{i2\pi f \frac{L}{2}} - e^{-i2\pi f \frac{L}{2}}}{2i} \right] = \frac{\sin \pi f L}{\pi f L}$$



13-41

$$\int_{-\infty}^{\infty} |f(t)| dt = \int_{-\infty}^{\infty} u(t) dt = \int_0^{\infty} 1 dt = \infty$$

\therefore Unit step function cannot have a F.T.

13-42

$$S_{FX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} F^*(i\omega) X(i\omega)$$

$$= \left[\lim_{T \rightarrow \infty} \frac{1}{2\pi T} F^*(i\omega) F(i\omega) \right] H(i\omega) = S_F(i\omega) H(i\omega)$$

$$S_{XF}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} X^* F = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} F^* H^* F = S_F^*(i\omega) H^*(i\omega)$$

$$\frac{S_{FX}}{S_{XF}} = \frac{S_F |H| e^{i\phi}}{S_F |H| e^{-i\phi}} = e^{i2\phi(\omega)}$$

13-42 Cont:

$$\frac{S_F}{S_{XF}} = \frac{S_X}{S_X H^*} \quad \frac{S_{FX}}{S_F} = \frac{S_X H}{S_X} = H$$

$$\frac{S_F}{S_{XF}} = \frac{1}{a-ib} = \frac{(a+ib)}{(a^2+b^2)} = \frac{H(i\omega)}{|H|^2}$$

13-43

$$\frac{d^2 \bar{u}(x,s)}{dx^2} = \left(\frac{s}{c}\right)^2 \bar{u}(x,s)$$

$$\bar{u}(x,s) = C_1 e^{\frac{sx}{c}} + C_2 e^{-\frac{sx}{c}}$$

$$\bar{F}(x,s) = AE \frac{d\bar{u}}{dx} = AE \frac{s}{c} [C_1 e^{\frac{sx}{c}} - C_2 e^{-\frac{sx}{c}}]$$

$$\bar{F}(0,s) = AE \frac{s}{c} [C_1 - C_2]$$

$$\bar{F}(l,s) = 0 = C_1 e^{\frac{sl}{c}} - C_2 e^{-\frac{sl}{c}} \quad \therefore C_1 = C_2 e^{-\frac{2sl}{c}}$$

$$C_2 = \frac{-c \bar{F}(0,s)}{AE s (1 - e^{-\frac{2sl}{c}})}$$

$$\begin{aligned} \bar{u}(x,s) &= \frac{-c \bar{F}(0,s)}{AE s (1 - e^{-\frac{2sl}{c}})} \left[e^{-\frac{2sl}{c}} e^{\frac{sx}{c}} + e^{-\frac{sx}{c}} \right] \\ &= \frac{-c \bar{F}(0,s) e^{-\frac{sl}{c}}}{AE s (1 - e^{-\frac{2sl}{c}})} \left[e^{\frac{s}{c}(x-l)} + e^{-\frac{s}{c}(x-l)} \right] \end{aligned}$$

13-44

$$p(x,t) = F_0 e^{i\omega t} \delta(x)$$

$$\bar{p}(x,s) = \frac{F_0}{s-i\omega} \delta(x), \quad \bar{F}(0,s) = \int_0^l \bar{p}(x,s) dx = \frac{F_0}{s-i\omega}$$

$$\bar{u}(x,s) = \frac{-c F_0 e^{\frac{sl}{c}}}{s(s-i\omega) AE (1 - e^{-\frac{2sl}{c}})} \left[e^{\frac{s}{c}(x-l)} + e^{-\frac{s}{c}(x-l)} \right]$$

$$\therefore u(x,t) = \frac{-c F_0 e^{i\omega t}}{\omega A E \sin \frac{\omega l}{c}} \cos \frac{\omega l}{c} \left(\frac{x}{l} - 1 \right)$$

$$\sigma = E \frac{du}{dx} = \frac{-c F_0 e^{i\omega t}}{\omega A \sin \frac{\omega l}{c}} \cdot \frac{\omega l}{c} \sin \frac{\omega l}{c} \left(\frac{x}{l} - 1 \right) = \frac{-F_0 e^{i\omega t}}{A \sin \frac{\omega l}{c}} \sin \frac{\omega l}{c} \left(\frac{x}{l} - 1 \right)$$

$$u(x, t) = \sum_{n=0}^{\infty} \phi_n(x) q_n(t)$$

$$\sigma(x, t) = E \frac{du}{dx} = E \sum_{n=1}^{\infty} \phi'_n(x) q_n(t) \quad \text{where } \phi' = \frac{d\phi}{dx}$$

$$\begin{aligned} \overline{\sigma(x, t) \sigma(x', t)} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \sigma(x, t) \sigma(x', t) dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \sigma^*(x, i\omega) \sigma(x', i\omega) d\omega \end{aligned}$$

$$\sigma(x, i\omega) = E \sum Q_n(i\omega) \phi'_n(x)$$

$$\overline{\sigma(x, t) \sigma(x', t)} = \frac{E^2}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \phi'_n(x) \phi'_k(x') \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi T} Q_n^*(i\omega) Q_k(i\omega) d\omega$$

$$\ddot{q}_n + \omega_n^2(1+i\gamma) q_n = \frac{1}{m\ell} \int_0^L p(x, t) \delta(x) \phi_n(x) dx = \frac{1}{m\ell} F(0, t) \phi_n(0)$$

$$Q_n(i\omega) = \frac{F(0, i\omega) \phi_n(0)}{m\ell[(\omega_n^2 - \omega^2) + i\gamma\omega_n^2]} = \frac{F(0, i\omega) \phi_n(0)}{m\ell\omega_n^2[1 - (\frac{\omega}{\omega_n})^2 + i\gamma]}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \frac{F^*(0, i\omega) F(0, i\omega) \phi_n(0) \phi_k(0)}{m^2 \ell^2 \omega_n^2 \omega_k^2 [1 - (\frac{\omega}{\omega_n})^2 - i\gamma][1 - (\frac{\omega}{\omega_k})^2 + i\gamma]} d\omega \\ &= \int_{-\infty}^{\infty} \frac{S(i\omega) \phi_n(0) \phi_k(0)}{m^2 \ell^2 \omega_n^2 \omega_k^2 [1 - (\frac{\omega}{\omega_n})^2 - i\gamma][1 - (\frac{\omega}{\omega_k})^2 + i\gamma]} d\omega \end{aligned}$$

Greatest contribution occurs when $k = n$ \therefore change double \sum to single summation with $k = n$.

13-45 Cont:

$$\overline{\sigma(x,t)\sigma(x',t)} = \frac{E^2}{2} \sum_n \phi_n'(x) \phi_n'(x') \phi_n^2(0) \frac{1}{m^2 l^2 \omega_n^4} \int_{-\infty}^{\infty} \frac{S(\omega) d\omega}{[1 - (\frac{\omega}{\omega_n})^2]^2 + \gamma^2}$$

$$\overline{\sigma^2(x,t)} = \frac{E^2}{2} \sum_n \phi_n^2(x) \phi_n^2(0) \frac{1}{m^2 l^2 \omega_n^4} S(\omega_n) \frac{\pi}{\gamma}$$

where:

$$\phi_n = \sqrt{2} \cos n\pi(\frac{x}{l} - 1)$$

$$\omega_n = n\pi \frac{C}{l}$$

$$C = \sqrt{\frac{AE}{m}}$$

$$\bar{E} = \frac{C^2 m}{A}$$

$$\therefore \overline{\sigma^2(x,t)} \approx \frac{2\pi}{\gamma} \sum \frac{C}{A^2 n\pi l} S(\omega_n) \sin^2 \frac{n\pi x}{l}$$

13-46

Prove $FT[x(t-t_0)] = e^{-i2\pi ft_0} X(f)$

where $X(f) = FT[x(t)]$

From Eq.(13.6-1) $x(t-t_0) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f(t-t_0)} df$
 $= \int_{-\infty}^{\infty} [e^{-i2\pi ft_0} X(f)] e^{i2\pi ft} df$

Comparison with Eq.(13.6-2) shows $e^{-i2\pi ft_0} X(f) = FT[x(t-t_0)]$

13-47

Prove $FT[x(t)*y(t)] = X(f)Y(f)$

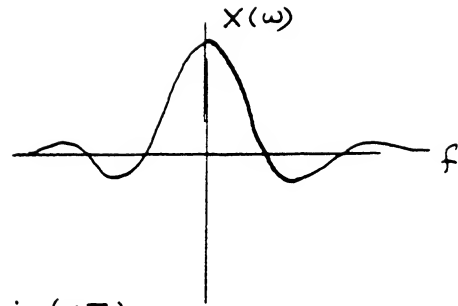
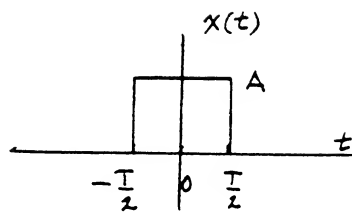
$x(t)*y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau = \text{convolution of } x(t) \text{ and } y(t)$

From Eq.(13.6-2)
 $FT[x(t)*y(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right] e^{-i2\pi ft} dt$
 $= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y(t-\tau) e^{-i2\pi ft} dt \right] x(\tau) d\tau$

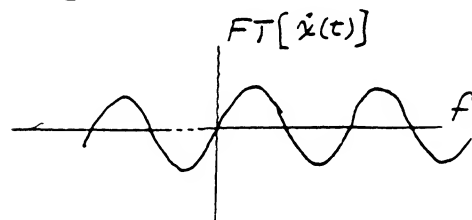
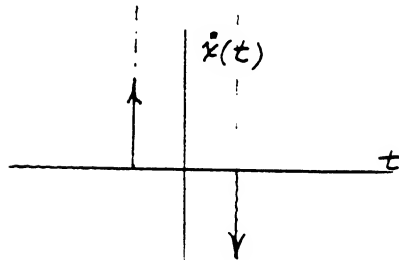
Let $(t-\tau) = \xi$, $t = \xi + \tau$ $dt = d\xi$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y(\xi) e^{-i2\pi f\xi} d\xi \right] x(\tau) e^{-i2\pi f\tau} d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau) e^{-i2\pi f\tau} d\tau \cdot \int_{-\infty}^{\infty} y(\xi) e^{-i2\pi f\xi} d\xi = X(f)Y(f)$$

13-48



$$X(\omega) = \mathcal{FT}[x(t)] = A T \frac{\sin(\frac{\omega T}{2})}{(\frac{\omega T}{2})}$$



$$\begin{aligned} \mathcal{FT}[\dot{x}(t)] &= i\omega \mathcal{FT}[x(t)] = i\cancel{\omega} \cdot A\cancel{T} \frac{\sin(\frac{\omega T}{2})}{\cancel{\omega T} \cancel{2}} \\ &= i 2A \sin(\frac{\omega T}{2}) = \text{sine wave} \end{aligned}$$

13-49

$$\int_{-\infty}^{\infty} \frac{a}{b+x^2} dx = 1 \quad (\text{Probability density function})$$

$$\begin{aligned} \text{(i) if } b > 0, \int_{-\infty}^{\infty} \frac{a}{b+x^2} dx &= \frac{a}{\sqrt{b}} \arctan \frac{x}{\sqrt{b}} \Big|_{-\infty}^{\infty} \\ &= \frac{a}{\sqrt{b}} \pi = 1 \Rightarrow a = \frac{\sqrt{b}}{\pi} \quad (b > 0) \end{aligned}$$

$$\text{(ii) if } b < 0, \frac{a}{x^2+b} = \frac{a}{2\sqrt{-b}} \left[\frac{-1}{x+\sqrt{-b}} + \frac{1}{x-\sqrt{-b}} \right]$$

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} P(x) dx &= \frac{a}{2\sqrt{-b}} \left[-\ln \left(\frac{x-\sqrt{-b}}{x+\sqrt{-b}} \right) \Big|_{-\infty}^{\infty} \right] \\ &= \frac{a}{2\sqrt{-b}} \left[\ln \left(\frac{x-\sqrt{-b}}{x+\sqrt{-b}} \right) \Big|_{-\infty}^{-\sqrt{-b}} + \ln \left(\frac{x-\sqrt{-b}}{x+\sqrt{-b}} \right) \Big|_{\sqrt{-b}}^{\infty} \right] \\ &= \frac{a}{2\sqrt{-b}} \left[\infty - 0 + 0 - (-\infty) \right] \longrightarrow \pm \infty \text{ if } a \neq 0 \end{aligned}$$

$$\text{if } a = 0, \int_{-\infty}^{\infty} P(x) dx = 0$$

\therefore Not a Probability density function if $b < 0$

$$\text{(iii) if } b = 0, \int_{-\infty}^{\infty} P(x) dx = \begin{cases} 0 & \text{if } a = 0 \\ \pm \infty & \text{if } a \neq 0 \end{cases}$$

\therefore Not a Probability density function

$$\therefore \text{ Answer } a = \frac{\sqrt{b}}{\pi} \quad (b > 0).$$

$$\frac{13-50}{p(x)} = \begin{cases} 1 - \exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases}, \quad \lambda > 0$$

$$\text{pdf: } p(x) = \frac{dp}{dx} = \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\text{mean: } \bar{x} = \int_0^{\infty} \lambda x \exp(-\lambda x) dx$$

$$= \left. \frac{-\lambda x - 1}{\lambda} e^{-\lambda x} \right|_0^{\infty} = \frac{1}{\lambda}$$

$$\begin{aligned} \text{variance: } \sigma^2 &= \int_0^{\infty} \lambda x^2 \exp(-\lambda x) dx \\ &= \left. -x^2 e^{-\lambda x} + \frac{2(-\lambda x - 1)}{\lambda^2} e^{-\lambda x} \right|_0^{\infty} \\ &= \frac{2}{\lambda^2} \end{aligned}$$

13-51

From problem 5-65

$$\ddot{x} - l\ddot{\theta} \sin\theta - l\dot{\theta}^2 \cos\theta + \frac{k}{m}x = 0$$
$$l\ddot{\theta} - \ddot{x} \sin\theta + g \sin\theta = 0$$

Neglecting the motion in the vertical x direction and linearizing the motion in the θ direction, we get

$$l\ddot{\theta} + g\theta = 0$$

Let the wind force be equal to $m\mu$. Then

$$l\ddot{\theta} + g\theta = \mu$$

Laplace transform $\Rightarrow (ls^2 + g)\theta(s) = \mu(s) \Rightarrow \theta(s) = \frac{1}{ls^2 + g} \mu(s)$

μ has white-noise characteristics.

Assume its mean and variance are W and σ^2 , respectively.

The mean of the output θ is

$$w_\theta = \left| \frac{1}{l(j\omega)^2 + g} \right| \cdot W = \frac{W}{g}$$

The variance of the output θ is

$$\sigma_\theta^2 = \left| \frac{1}{l(j\omega)^2 + g} \right|^2 \cdot \sigma^2 = \frac{\sigma^2}{g^2}$$

Assuming Gaussian distribution, the pdf for θ is

$$p(\theta) = \frac{1}{\sqrt{2\pi} \sigma_\theta} \exp\left[-\frac{1}{2} \left(\frac{\theta - w_\theta}{\sigma_\theta}\right)^2\right]$$

$$\therefore P[\theta \leq 10^\circ = \frac{\pi}{18}] = \int_{-\infty}^{\pi/18} \frac{1}{\sqrt{2\pi} \sigma_\theta} \exp\left[-\frac{1}{2} \left(\frac{\theta - w_\theta}{\sigma_\theta}\right)^2\right] d\theta$$
$$= \frac{1}{\sigma_\theta} \left[\operatorname{erf}\left(\frac{\pi/18 - w_\theta}{\sigma_\theta}\right) - \operatorname{erf}\left(\frac{-\infty - w_\theta}{\sigma_\theta}\right) \right],$$

where, $\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp(-\frac{1}{2}x^2) dx$

14-1 If $x_1 = \varphi_1(t)$ and $x_2 = \varphi_2(t)$ are solutions of the equation $\ddot{x} + x^3 = 0$, then they will satisfy

$$\ddot{\varphi}_1 + \varphi_1^3 = 0 \quad \text{and} \quad \ddot{\varphi}_2 + \varphi_2^3 = 0$$

Adding, the following is also true

$$(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (\varphi_1^3 + \varphi_2^3) = 0 \quad (a)$$

If we assume $x = \varphi_1 + \varphi_2$ and substitute into the D.E. we would obtain

$$(\ddot{\varphi}_1 + \ddot{\varphi}_2) + (\varphi_1^3 + 3\varphi_1^2\varphi_2 + 3\varphi_1\varphi_2^2 + \varphi_2^3) = 0$$

which does not agree with the correct result (a)

Superposition of solutions are in general not solutions of nonlinear equations.

14-2 $\Sigma F_x = -2T \sin \theta = m\ddot{x}$

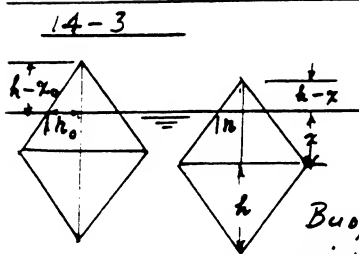
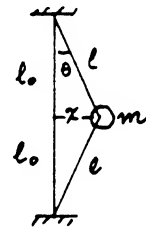
$$\sin \theta = \frac{x}{l} = \frac{x}{\sqrt{l_0^2 + x^2}} \approx \frac{x}{l_0} \left[1 - \frac{1}{2} \left(\frac{x}{l_0} \right)^2 \right]$$

$$T = T_0 + k(l - l_0) = T_0 + k \left[l_0 \left(1 + \frac{x^2}{2l_0^2} \right) - l_0 \right]$$

$$\approx T_0 + k \frac{1}{2} \left(\frac{x}{l_0} \right)^2$$

$$\therefore m\ddot{x} + 2 \left[T_0 + \frac{k}{2} \left(\frac{x}{l_0} \right)^2 \right] \frac{x}{\sqrt{l_0^2 + x^2}} = 0$$

$$m\ddot{x} + \frac{2}{l_0} \left[T_0 + \frac{k}{2} \left(\frac{x}{l_0} \right)^2 \right] \left[1 - \frac{1}{2} \left(\frac{x}{l_0} \right)^2 \right] x = 0$$



Vol. of cone = $\frac{1}{3} \pi r^2 (h-x)$

From similar triangles $\frac{r_0}{h-x_0} = \frac{r}{h-x}$

$$\therefore r = r_0 \left(\frac{h-x}{h-x_0} \right)$$

Difference in vol. = $\frac{1}{3} \pi [r^2 (h-x) - r_0^2 (h-x_0)]$

Buoyant force = $\rho \Delta V = \frac{\pi}{3} \rho r_0^2 \left[\left(\frac{h-x}{h-x_0} \right)^2 (h-x) - (h-x_0) \right]$
= weight of water displaced.

$$\therefore m\ddot{x} = \frac{\pi}{3} \rho r_0^2 \left[\frac{(h-x)^3}{(h-x_0)^2} - (h-x_0) \right]$$

$$= \frac{\pi}{3} \rho \frac{r_0^2}{(h-x_0)^2} \left[(h-x)^3 - (h-x_0)^3 \right]$$

14-4 For $x > x_0$ Eq. of motion is

$$m\ddot{x} + k(x - x_0) = 0 \quad \dots \dots \dots (a)$$

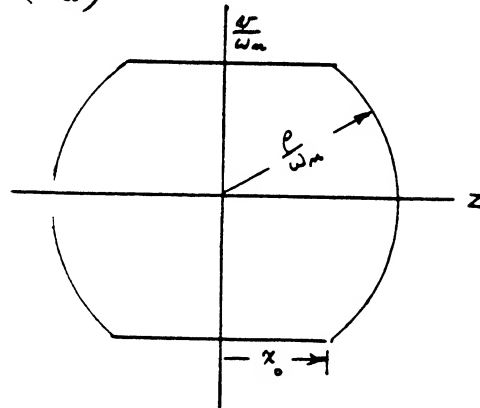
For $x < x_0$ $\ddot{x} = \dot{x} \frac{d\dot{x}}{dx} = 0 \quad \therefore \frac{dv}{dx} = 0$ where $v = \dot{x}$

Let $z = (x - x_0)$ in Eq. (a); then

$$\ddot{z} + \omega_m^2 z = 0 \quad \text{or} \quad \dot{z} \frac{d\dot{z}}{dz} + \omega_m^2 z = 0$$

Integrating $\dot{z}^2 + \omega_m^2 z^2 = \left(\frac{v}{\omega_m}\right)^2$ a circle

Phase-plane trajectory
will look like this



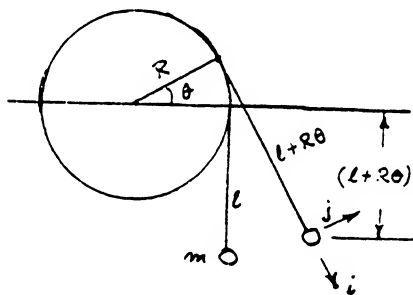
14-5

Since system is conservative

$$\frac{d}{dt}(T + U) = 0 \quad (a)$$

Vel. of m is

$$\vec{v} = (l + R\dot{\theta})\dot{\theta}\vec{j}$$



$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m \dot{\theta}^2 (l + R\dot{\theta})^2$$

$$U = mg[l - (l + R\dot{\theta})\cos\theta + R\sin\theta]$$

Subst. into (a)

$$\dot{\theta} \left\{ \ddot{\theta} (l + R\dot{\theta})^2 + \dot{\theta} (l + R\dot{\theta}) R + g(l + R\dot{\theta}) \sin\theta - R\cos\theta + R\sin\theta \right\} = 0$$

$$\ddot{\theta} + \frac{R\dot{\theta}^2}{l + R\dot{\theta}} + \frac{g}{l + R\dot{\theta}} \sin\theta = 0$$

14-6

$$\ddot{x} + \omega_m^2 x = 0$$

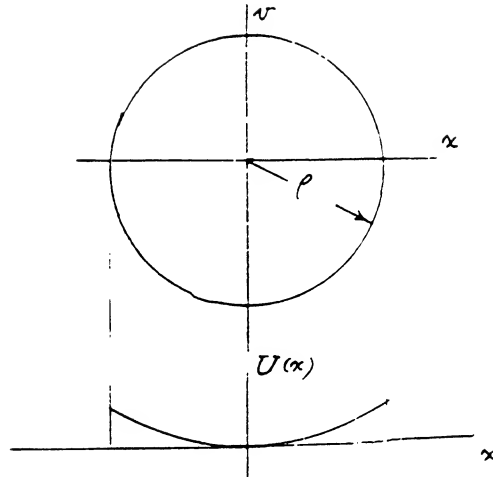
$$\ddot{x} = \dot{x} \frac{dx}{dx}, \text{ Let } v = \frac{\dot{x}}{\omega_m}$$

then above eq. becomes

$$\frac{dv}{dx} = -\frac{v}{x} \text{ or } v^2 + x^2 = \rho^2$$

phase-plane traj. is a circle

$$U = \frac{1}{2} k x^2 \text{ a parabola}$$



14-7

$$\gamma = 4 \int_0^{x_{\max}} \frac{dx}{\sqrt{2[E - U(x)]}}$$

$$U(x) = \frac{1}{2} \frac{k}{m} x^2 \text{ per unit mass}$$

$$E = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \frac{k}{m} x^2$$

$$\dot{x} = 0 \text{ when } x = x_{\max} \therefore x_{\max} = \sqrt{\frac{2Em}{k}}$$

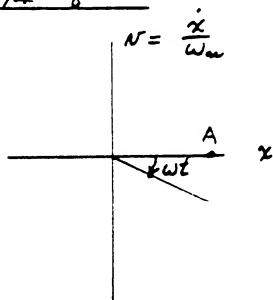
$$\gamma = 4 \int_0^{\sqrt{\frac{2Em}{k}}} \frac{dx}{\sqrt{2[E - \frac{1}{2} \frac{k}{m} x^2]}}$$

$$\text{but } \frac{k}{m} x^2 = \omega_m^2 x^2$$

$$C^2 = 2E$$

$$\gamma = \frac{4}{\omega_m} \int_0^{u = \omega_m x_{\max} = C} \frac{du}{\sqrt{C^2 - u^2}} = \frac{4}{\omega_m} \sin^{-1}\left(\frac{u}{C}\right) \Big|_0^C = \frac{4}{\omega_m} \frac{\pi}{2} = \frac{2\pi}{\omega_m}$$

14-8



$$x(0) = A$$

$$\dot{x}(0) = 0$$

$$\text{Let } y = \dot{x}$$

$$\dot{y} = -\omega_m^2 x$$

$$V = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{y^2 + \omega_m^4 x^2} = 0$$

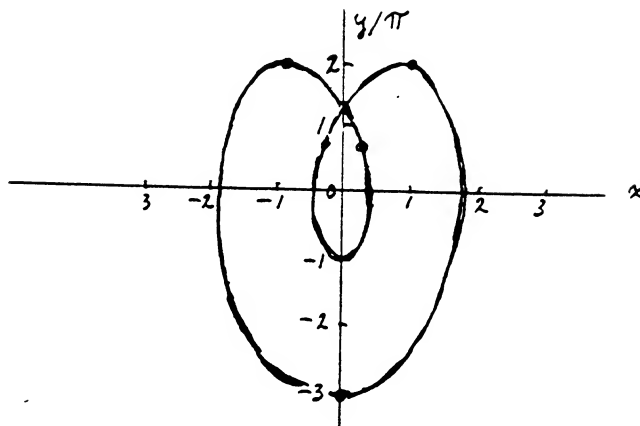
$$V = 0 \text{ only if } x = y = 0$$

14-9

$$x = \cos \pi t + \sin 2\pi t$$

$$y = \dot{x} = -\pi \sin \pi t + 2\pi \cos 2\pi t$$

t	x	y
0	1	2π
.25	-.3	.7π
.50	0	-π
.75	.3	.7π
1.0	-1	2π
1.25	-1.7	-.7π
1.5	0	-3π
1.75	1.7	-.7π
2.0	1	2π



14-10

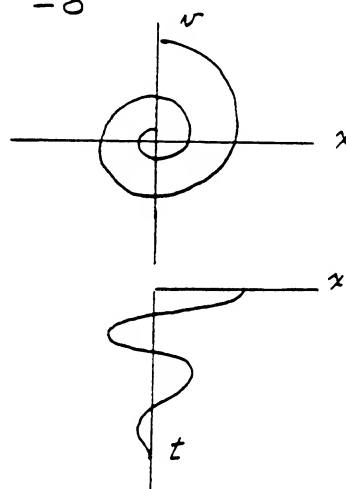
$$\ddot{x} + 25\omega_m \dot{x} + \omega_m^2 x = 0$$

$$\dot{x} \frac{d\dot{x}}{dx} = -25\omega_m \dot{x} - \omega_m^2 x$$

$$\frac{1}{\omega_m} \frac{d\dot{x}}{dx} = -25 - \omega_m \frac{x}{\dot{x}}$$

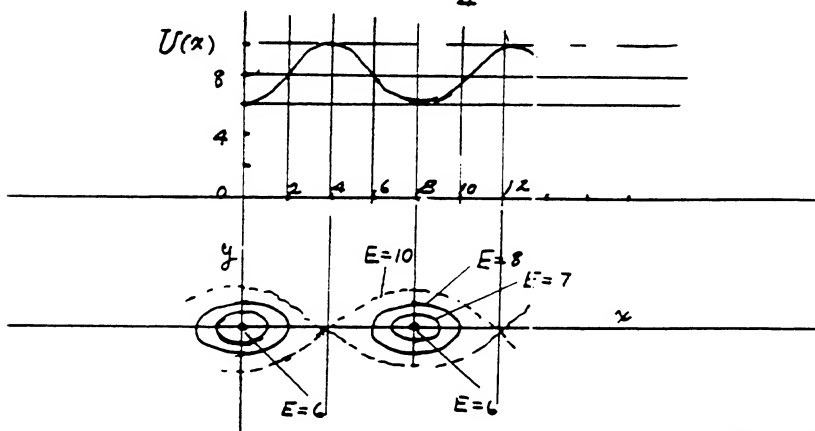
$$\text{Let } v = \frac{\dot{x}}{\omega_m}$$

$$\frac{dv}{dx} = -25 - \frac{x}{v} \quad \text{Trajectory is a spiral.}$$



14-12

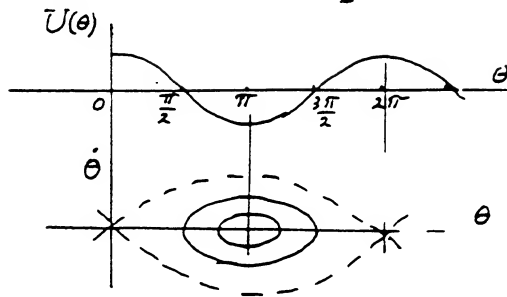
$$U = 8 - 2 \cos \frac{\pi x}{4}$$



see next page for Prob 14-11

14-11

$$U = \frac{g}{L} \cos \theta$$



The origin of phase plane is shifted to π , as compared to Fig 14.4-2

$\theta = 0$ and 2π are unstable points

14-13

$$\frac{dy}{dx} = \frac{2x+2y}{5x-y} = \frac{P}{Q}$$

singular points are $\frac{P}{Q} = \frac{0}{0}$

$$2x+2y = 0$$

$$5x-y = 0$$

$$12x = 0$$

$$\therefore x = 0, y = 0$$

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{Eq. 14.3-8}$$

$(a+e) > 0 \quad \therefore$ system is unstable and aperiodic

$$\lambda_1 = 3.0 \quad u = e^{3t}$$

$$\lambda_2 = 4.0 \quad v = e^{4t}$$

14-14

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\begin{vmatrix} 5-\lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\text{gives } \lambda = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$

subst λ into eq.

$$(5-\lambda)x = y \quad \text{gives } x^{(1)} = 0.5 y^{(1)}$$

$$x^{(2)} = 1.0 y^{(2)}$$

$$P = \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix}$$

Transformation to decouple is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

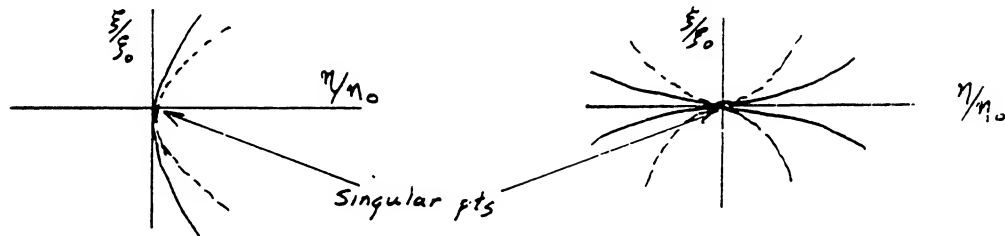
$$\therefore \dot{\xi} = 3\xi \quad \text{and} \quad \dot{\eta} = 4\eta \quad \text{decoupled eqs.}$$

14-15 Uncoupled eqs. $\begin{cases} \dot{\xi} = \lambda_1 \xi \\ \dot{\eta} = \lambda_2 \eta \end{cases}$ can be written as

$$\frac{d\xi}{d\eta} = \left(\frac{\lambda_1}{\lambda_2}\right) \frac{\xi}{\eta} \quad \text{Integrating} \quad \xi = \xi_0 \left(\frac{\eta}{\eta_0}\right)^{\frac{\lambda_1}{\lambda_2}}$$

For $\frac{\lambda_1}{\lambda_2} = .5$, the traj. are tangent to ξ as shown

If $\frac{\lambda_1}{\lambda_2} = 2$, the traj. are tangent to η



14-16 $\frac{d\xi}{d\eta} = 2 \frac{\xi}{\eta} \quad \therefore \xi = \xi_0 \left(\frac{\eta}{\eta_0}\right)^2$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [P] \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = [P] \begin{Bmatrix} \xi_0 \left(\frac{\eta}{\eta_0}\right)^2 \\ \eta \end{Bmatrix}$$

\therefore Eq for u & v are of the form

$$\left. \begin{aligned} u &= A\eta^2 + B\eta = x - x_s \\ v &= C\eta^2 + D\eta = y - y_s \end{aligned} \right\} \begin{array}{l} \text{only a Linear} \\ \text{shift of origin} \end{array}$$

Need original eq.

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{corresponding to } \frac{\lambda_1}{\lambda_2} = 2$$

a, b, c, e must be known before plotting.

Use Eq. 12.3-9

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = P \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

From Eq. 12.3-8

$$\begin{vmatrix} (a-\lambda) & b \\ c & (e-\lambda) \end{vmatrix} = 0 \quad \therefore \quad \frac{u_1}{v_1} = \frac{-b}{a-\lambda_1}$$

$$\frac{u_2}{v_2} = \frac{-b}{a-\lambda_2}$$

14-16 Cont. Since only relative values of u, v are essential, Let $v_1 = v_2 = 1.0$ and $u_1 = \frac{-b}{a-\lambda_1}, u_2 = \frac{-b}{a-\lambda_2}$
Using Eq. 14.3-8 we have

$$P \begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} P \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\xi} \\ \dot{\eta} \end{Bmatrix} = P^{-1} \begin{bmatrix} a & b \\ c & e \end{bmatrix} P \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = [\Lambda] \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$\therefore P^{-1} \begin{bmatrix} a & b \\ c & e \end{bmatrix} P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ where } P = \begin{bmatrix} \frac{-b}{a-\lambda_1} & \frac{-b}{a-\lambda_2} \\ 1 & 1 \end{bmatrix}$$

Equating the two elements on the two sides of the above eq.

$$\left. \begin{aligned} (au_1 + b) - (cu_1 + e) &= 0 \\ [-u_1(au_1 + b) + u_1(cu_1 + e)] &= 0 \\ \frac{1}{(u_1 v_2 - u_2 v_1)} [(au_1 + b) - (cu_1 + e)] &= \lambda_1 \\ \frac{1}{(u_1 v_2 - u_2 v_1)} [-u_1(au_1 + b) + u_1(cu_1 + e)] &= \lambda_2 \end{aligned} \right\} \begin{aligned} &\text{These 4 eqs. can be solved} \\ &\text{for } a, b, c, e, \text{ then subst.} \\ &\text{into Eq. 14.3-9} \\ &\text{Other alternative is to} \\ &\text{solve for } u \text{ \& } v \text{ from} \\ &\text{Eq. 14.3-12 for chosen values} \\ &\text{of } t. \end{aligned}$$

14-17 From given eq.

$$\begin{Bmatrix} \dot{v} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{Bmatrix} v \\ u \end{Bmatrix} \quad \begin{vmatrix} (\alpha - \lambda) & -\beta \\ -\beta & (\alpha - \lambda) \end{vmatrix} = 0$$

$$\therefore \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0 \quad \text{and} \quad \lambda = -\alpha \pm i\beta$$

14-18 $u = p \cos \theta \quad v = p \sin \theta$

$$du = dp \cos \theta - p \sin \theta d\theta \quad dv = dp \sin \theta + p \cos \theta d\theta$$

Subst. u & v in Eq. for Prob. 14-17

$$\frac{dv}{du} = \frac{\beta p \cos \theta + \alpha p \sin \theta}{\alpha p \cos \theta - \beta p \sin \theta} = \frac{dp \sin \theta + p \cos \theta d\theta}{dp \cos \theta - p \sin \theta d\theta}$$

$$\begin{aligned} (dp \sin \theta + p \cos \theta d\theta)(\alpha \cos \theta - p \sin \theta) &= (dp \cos \theta - p \sin \theta d\theta)(\beta \cos \theta + \alpha \sin \theta) \\ \frac{dp}{p} [\alpha \cos^2 \theta - \beta \sin^2 \theta - \alpha \sin \theta \cos \theta] &= -d\theta [\alpha \cos^2 \theta - \beta \sin^2 \theta + \beta \sin \theta \cos \theta + \alpha \sin^2 \theta] \end{aligned}$$

$$\therefore \frac{dp}{p} = \frac{\alpha}{\beta} d\theta \quad p = e^{\frac{\alpha}{\beta} \theta}$$

14-19

Eq. in x, y plane is $xy = \pm C$

node point at origin is unstable

$$x dy + y dx = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\begin{Bmatrix} \dot{y} \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} y \\ x \end{Bmatrix} \quad \left| \begin{array}{cc} -(1+\lambda) & 0 \\ 0 & (1-\lambda) \end{array} \right| = 0$$

$$(1-\lambda)(1+\lambda) = 0 \quad \lambda = \pm 1 \quad \begin{aligned} \xi &= \xi_0 e^t \\ \eta &= \eta_0 e^{-t} \end{aligned}$$

$$\frac{d\xi}{d\eta} = \frac{\lambda_1 \xi}{\lambda_2 \eta} = -\frac{\xi}{\eta} \quad \text{or} \quad \eta d\xi + \xi d\eta = 0$$

$$\therefore \xi \eta \text{ plot is } \xi \eta = \pm \text{const.} \quad \therefore \text{same}$$

14-20

The form of the eq. is

$$\frac{dv}{du} = \frac{v+u}{u}$$

$$\begin{Bmatrix} \dot{v} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ u \end{Bmatrix}$$

characteristic eq.

$$\left| \begin{array}{cc} (1-\lambda) & 1 \\ 0 & (1-\lambda) \end{array} \right| = 0 \quad \text{Leads to two equal roots } \lambda = 1$$

Transformation eq. 14.3-9 cannot be applied

14-21

Differentiate the eq. $x^2 + 2xy + 3y^2 = C$

$$2x dx + 2x dy + 2y dx + 6y dy = 0$$

$$(2x + 6y) dy = -(2x + 2y) dx \quad \therefore \quad \frac{dy}{dx} = \frac{-x-y}{x+3y}$$

14-22

Let $v = \dot{x}$ then the eq. $\ddot{x} + 25\omega_n \dot{x} + \omega_n^2 x = 0$

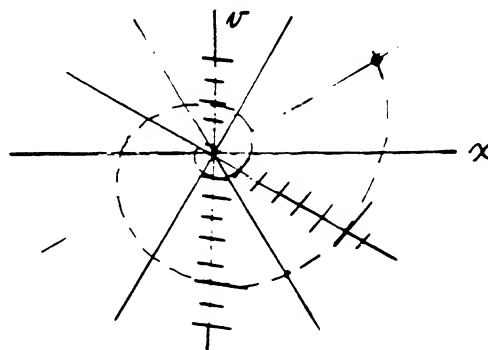
becomes $\dot{v} + 25\omega_n v + \omega_n^2 x = 0$, $v \frac{dv}{dx} + 25\omega_n v + \omega_n^2 x = 0$

Let $\frac{dv}{dx} = \text{constant for isocline} = C$,

$$v(C + 25\omega_n) + \omega_n^2 x = 0$$

= straight line through origin

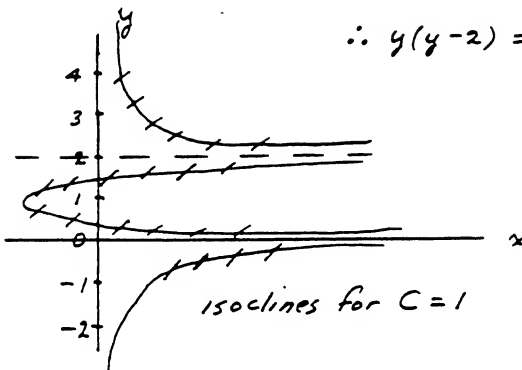
As $t \rightarrow \infty$ the points on the isoclines move towards the origin \therefore stable



14-23

$$\frac{dy}{dx} = xy(y-2) = \text{constant for isocline}$$

$$\therefore y(y-2) = \frac{C}{x}$$



y	y(y-2)	x for C=1
-3	15	.066
-2	8	.125
-1	3	.333
0	0	∞
1	-1	-1
2	0	∞
3	3	.333
4	8	.125

14-24

$$y \frac{dy}{dx} + \omega_m^2 x + \mu x^3 = 0$$

$$\text{integrating } y^2 + \omega_m^2 x^2 + \frac{1}{2} \mu x^4 = 2E$$

$$\text{or } 2T + 2U = 2E$$

$$\text{Since } y = \frac{dx}{dt} \quad dt = \frac{dx}{y} = \frac{dx}{\sqrt{2E - 2U}}$$

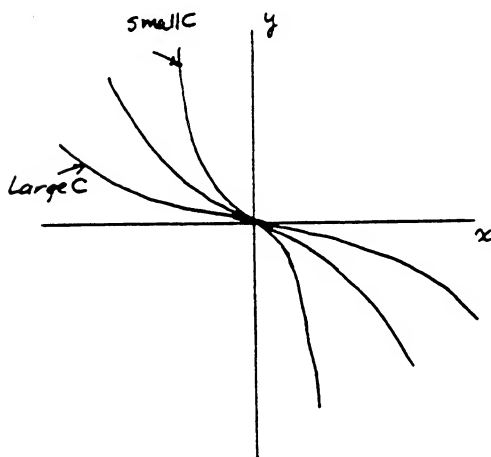
$$\text{For } \frac{1}{4} \text{ period} \quad \frac{T}{4} = \int_0^A \frac{dx}{\sqrt{2(E - U)}}$$

14-25

From Prob. 14-24

$$\frac{dy}{dx} = \frac{-\omega_m^2 x - \mu x^3}{y} = C$$

$$\text{Let } \frac{\omega_m^2}{C} = 4 + \frac{\mu}{C} = 2$$



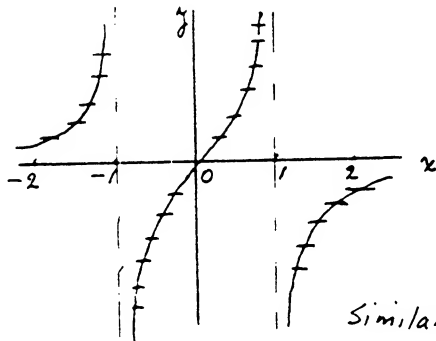
x	4+2x^2	y = -x(4+2x^2)
-1	6	6
0	4	0
1	6	-6
2	12	-24
3	22	-66

14-26 $\ddot{x} - \mu \dot{x}(1-x^2) + x = 0$

$y \frac{dy}{dx} - \mu y(1-x^2) + x = 0$

$\frac{dy}{dx} = \frac{\mu y(1-x^2) - x}{y} = \mu(1-x^2) - \frac{x}{y} = \text{const}$

For $\frac{dy}{dx} = C = 0$ and $\mu = 2$



x	y
0	0
±.2	±.104
±.4	±.239
±.6	±.45
±.8	±1.11
±.9	±2.37
±1.0	±∞
±2	∓.333
±4	∓.13

Similarly solve for $C = \pm 1$ and other values

14-27 Let $\omega_n^2 = \frac{g}{m}$ $\tau = \omega_n t$ $y = \frac{dx}{d\tau}$

Then $\dot{x} = \omega_n y$ $\ddot{x} = \omega_n^2 y \frac{dy}{dx}$

the eq $\ddot{x} + \frac{c}{m} \dot{x} + \frac{g}{m} x + \frac{\mu}{m} x^3 = 0$ becomes

$y \frac{dy}{dx} + \frac{c}{m} y + x + \frac{\mu}{\omega_n^2 m} x^3 = 0$

$\frac{dy}{dx} = \frac{-[x + (\frac{c}{m} y + \frac{\mu}{\omega_n^2 m} x^3)]}{y} = \alpha$

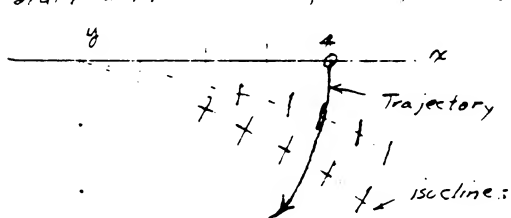
14-28 Above eq. can be written as given: $\omega_n^2 = 25$, $\frac{\mu}{m} = 5$

$(\alpha + \frac{c}{m}) y = -(x + \frac{\mu}{\omega_n^2 m} x^3)$ where $\alpha = \frac{dy}{dx}$ $\frac{c}{m} = 2.0$

$\therefore (\alpha + 2) y = -(x + 0.2 x^3)$ Assign different values for α and plot

the field of isoclines. From prob. 14-27 $\dot{x}(0) = \omega_n y(0) = 0$

\therefore Start with $x(0) = 4$, $\dot{x}(0) = 0 \therefore y(0) = 0$ and fill in trajectory



All isoclines are vertical along x axis. The right side of eq. is straight line + cubic which must be \div by $(\alpha + 2)$

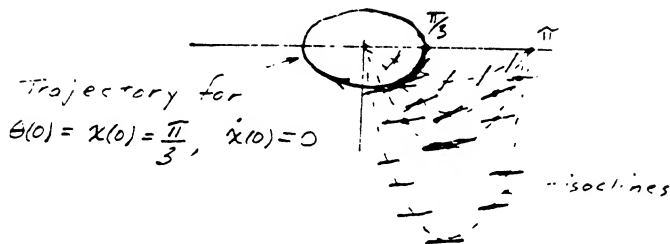
14-29 $\ddot{\theta} + \frac{g}{l} \sin \theta = 0$ Let $x = \theta$, $\omega_0^2 = g/l$

$y = \frac{\dot{\theta}}{\omega_0} = \frac{\dot{x}}{\omega_0} = \frac{dx}{d\tau}$
 $\therefore \omega_0^2 y \frac{dy}{dx} + \frac{g}{l} \sin x = 0$ with $\alpha = \frac{dy}{dx} = \text{constant}$

$\alpha y = -\left(\frac{g}{l\omega_0^2}\right) \sin x$ or $y = -\frac{g}{\alpha l\omega_0^2} \sin x = \text{eq. for isocline}$
 with $\alpha = \text{const.}$

Since $\frac{g}{l\omega_0^2} = 1$

$y = -\frac{1}{\alpha} \sin x = \text{isocline}$



14-30 $\ddot{\theta} + \frac{g}{l} \sin \theta = 0$ $\dot{\theta} \frac{d\dot{\theta}}{d\theta} + \frac{g}{l} \sin \theta = 0$

Integrate $\frac{\dot{\theta}^2}{2} - \frac{g}{l} \cos \theta = E$ $\therefore U = -\frac{g}{l} \cos \theta$

At $t=0$ $\theta = 60^\circ$ $\dot{\theta} = 0$ $\therefore 0 - \frac{g}{l} \cos 60^\circ = E$

Since $\dot{\theta} = \frac{d\theta}{dt}$ $dt = \frac{d\theta}{\dot{\theta}}$ $E = -\frac{g}{2l} = -\frac{g}{l} \cos \theta_0$

$dt = \frac{d\theta}{\sqrt{2E + 2\frac{g}{l} \cos \theta}}$ $t = \int \frac{d\theta}{\sqrt{2(E + \frac{g}{l} \cos \theta)}}$

$t = \sqrt{\frac{l}{g}} \int \frac{d\theta}{\sqrt{2(\cos \theta - \cos \theta_0)}}$ $\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$
 $\cos \theta - \cos \theta_0 = 2(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2})$

Let $\sin \frac{\theta}{2} = \sin \frac{\theta_0}{2} \sin \phi = k \sin \phi$ (i)

then $\cos \theta - \cos \theta_0 = 2 \sin^2 \frac{\theta_0}{2} (1 - \sin^2 \phi) = 2 \sin^2 \frac{\theta_0}{2} \cos^2 \phi$ (ii)

Diff (i) $\frac{1}{2} \cos \frac{\theta}{2} d\theta = k \cos \phi d\phi$ $\therefore d\theta = \frac{2k \cos \phi d\phi}{\cos \frac{\theta}{2}}$ (iii)

Subst (ii) & (iii) into t

$t = \sqrt{\frac{l}{g}} \int \frac{2k \cos \phi d\phi}{\cos \frac{\theta}{2} 2 \sin^2 \frac{\theta_0}{2} \cos^2 \phi} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\cos \frac{\theta}{2}}$

14-30 Cont:

$$t = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

$$\therefore \text{period } \tau = 4 \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad k = \sin \frac{\theta_0}{2}$$

Since $\sin \phi = \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}}$, when $\theta = 0$, $\phi = 0$
 when $\theta = \theta_0$, $\phi = \frac{\pi}{2} = \frac{1}{4} \text{ cycle}$

14-31 $\ddot{x} + \omega_n^2 x + C \operatorname{sgn}(\dot{x}) = 0 \quad \tau = \omega_n t$

$$\dot{x} \frac{d\dot{x}}{dx} + \omega_n^2 x + C \operatorname{sgn}(\dot{x}) = 0 \quad \text{Let } \frac{dx}{d\tau} = y = \frac{\dot{x}}{\omega_n}$$

$$\omega_n^2 y \frac{dy}{dx} + \omega_n^2 x + C \operatorname{sgn}(y) = 0$$

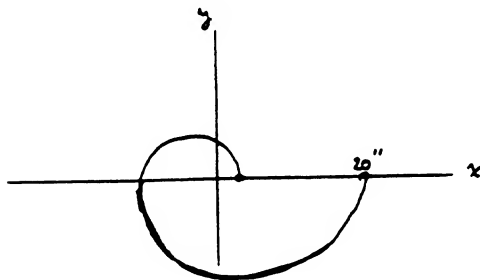
$$\frac{dy}{dx} = - \frac{\frac{1}{\omega_n} C \operatorname{sgn}(y) + x}{y} = - \frac{f(y) + x}{y}$$

where $f(y) = \frac{1}{\omega_n} C \operatorname{sgn}(y)$

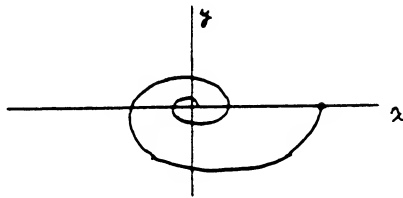
14-32 Initial values $x(0) = 20^\circ \quad y(0) = 0$

$$\omega_n = \sqrt{\frac{3.60}{.10}} = 6, \quad \mu = 0.20$$

$$\frac{\mu g}{\omega_n^2} = \frac{.20 \times 386}{36} = 2.145 \text{ in}$$



14-33 For the undamped pendulum the trajectory is an ellipse, For the damped pendulum the curve is inside of the ellipse, as shown



14-34 $\ddot{\theta} + \omega_n^2 \sin \theta = 0$ $\sin \theta \approx \theta - \frac{\theta^3}{6}$
 $\ddot{\theta} + \omega_n^2 (\theta - \frac{\theta^3}{6}) = 0$

Let $\theta = \theta_0 + \mu \theta_1$, and $\omega^2 = \omega_n^2 + \mu \alpha$

From Eq. 12.6-9

$$\omega^2 = \omega_n^2 + \frac{3}{4} \mu A^2 = \omega_n^2 \left[1 + \frac{3}{4} \times \frac{1}{6} \theta_0^2 \right]$$

$$\therefore \omega = \omega_n \sqrt{1 + \frac{1}{8} \theta_0^2} \approx \omega_n \left(1 + \frac{1}{16} \theta_0^2 \right) \quad \omega_n = \sqrt{\frac{g}{L}}$$

14-35 From Prob 14-34

$$\omega = \frac{2\pi}{T} \approx \frac{2\pi}{T_n} \left(1 + \frac{1}{16} \theta_0^2 \right) \therefore T \approx T_n \left(\frac{1}{1 + \frac{1}{16} \theta_0^2} \right)$$

14-36 $\ddot{x} + 0.15 \dot{x} + 10x + x^3 = 5 \cos(\omega t + \phi)$

$$\omega_n^2 = 10, \quad c = .15, \quad \mu = 1.0, \quad F = 5$$

Eq. 4.6-11 becomes

$$25 = \left[(10 - \omega^2)A + \frac{3}{4} A^3 \right]^2 + [.15 \omega A]^2$$

Rearrange to

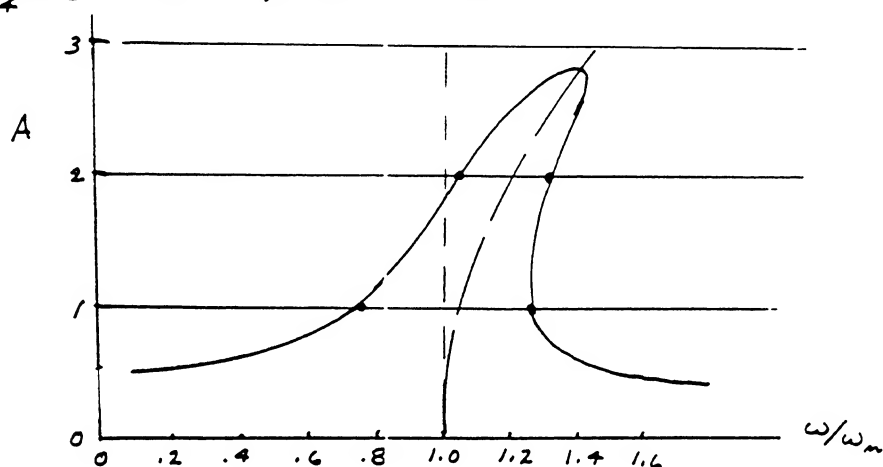
$$\omega^4 - \left(20 + \frac{3}{2} A^2 - .0225 \right) \omega^2 + \left(100 + 15 A^2 + \frac{9}{16} A^4 - \frac{25}{A^2} \right) = 0$$

$$\omega^4 - b \omega^2 + c = 0$$

14-36 Cont.:

A	b	c	ω^2	ω/ω_m
0	19.98	$-\infty$	∞	∞
1	21.48	90.5	$\begin{cases} 5.7 \\ 15.78 \end{cases}$	$\begin{cases} .755 \\ 1.255 \end{cases}$
2	25.98	162.7	$\begin{cases} 10.69 \\ 15.29 \end{cases}$	$\begin{cases} 1.036 \\ 1.24 \end{cases}$
3	32.48	272.7	complex	this indicates the peak to be below $A=3$
$\approx .5$				0

$$\sqrt{\frac{b^2}{4} - c} = 0 \quad \text{gives } A \approx 2.9$$



14-37

$$\frac{\omega/\omega_m \text{ vs } \phi_1 \text{ \& } \phi_2}{\phi_1 = \text{branch 1} \quad \phi_2 = \text{'' 2}}$$

$$\tan \phi = \frac{B_0}{A_0} = \frac{c\omega}{(\omega_m^2 - \omega^2) + \frac{3}{4}\mu A^2} \quad \text{Eq. 14.6-10}$$

$$= \frac{.15\omega}{(10 - \omega^2) + \frac{3}{4}A^2}$$

subst. ω & A from Prob. 14-36

$$(\tan \phi)_{A=1} = \frac{.15 \sqrt{5.7}}{(10 - 5.7) + .75} = \frac{.358}{5.05} = .071, \quad \phi = 4^\circ 4'$$

14-37 Cont:

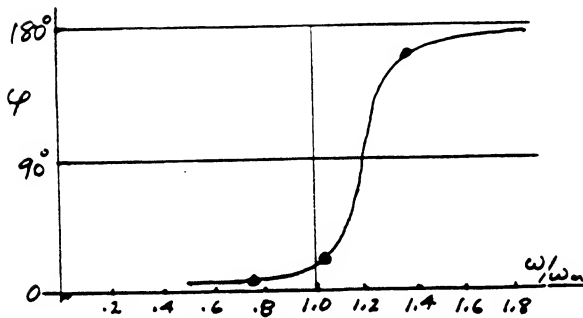
$$(\tan \phi)_{A=1} = \frac{.15 \sqrt{15.78}}{(10 - 15.78) + .75} = \frac{.60}{-5.03} = -.119 \quad \phi = 173^\circ 12'$$

$$(\tan \phi)_{A=2} = \frac{.15 \sqrt{10.69}}{(10 - 10.69) + 3} = \frac{.490}{2.31} = .213 \quad \phi = 12^\circ$$

$$= \frac{.15 \sqrt{15.29}}{(10 - 15.29) + 3} = \frac{.590}{-2.29} = -.257 \quad \phi = 185^\circ 35'$$

$$(\tan \phi)_{A=0} = \frac{\infty}{-\infty^2} = -\frac{1}{\infty} = -0 \quad \phi = 180^\circ -$$

$$(\tan \phi)_{A=5} = \frac{0}{(10 - 0) + .375} = 0 \quad \phi = 0^\circ$$



ω/ω_n	ϕ_1	ϕ_2
0	0	
.755	4° 4'	
1.036	12°	
1.25		173° 12'
1.24		185° 35'
∞		180°

14-38 Moment eq. about an accelerating point A is

$$\vec{M}_A = I_A \vec{\omega} + \vec{r}_{AC} \times m \vec{a}_A \quad \left(\begin{array}{l} \text{see Dynamics by} \\ \text{Pestel \& Thomson p 213} \\ \text{McGraw Hill.} \end{array} \right)$$

where C is the center of mass and \vec{r}_{AC} is a vector from A to C. For this problem $y_A = y_0 \cos 2\omega t$

$$I_A = ml^2 \quad |\vec{r}_{AC}| = l \quad |a_A| = -4y_0 \omega^2 \cos 2\omega t$$

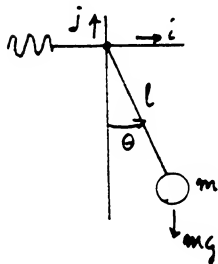
$$\vec{r}_{AC} \times m \vec{a}_A = l (\sin \theta \vec{i} - \cos \theta \vec{j}) \times m (-4y_0 \omega^2 \cos 2\omega t) \vec{j}$$

$$= -4y_0 \omega^2 l m \cos 2\omega t \cdot \vec{k}$$

$$\vec{\omega} = \ddot{\theta} \vec{k} \quad \vec{M}_A = -mgl \sin \theta \vec{k}$$

$$\therefore -mgl \sin \theta = ml^2 \ddot{\theta} - 4y_0 \omega^2 l m \cos 2\omega t$$

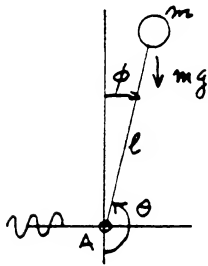
$$\ddot{\theta} + \left(\frac{g}{l} - \frac{4y_0 \omega^2 \cos 2\omega t}{l} \right) \sin \theta = 0$$



14-39

For the inverted pendulum $\theta = \pi - \phi$

$$\sin \theta = \sin \phi$$



$$\therefore -\ddot{\phi} + \left(\frac{g}{l} - \frac{4\omega_0^2 y_0}{l} \cos 2\omega t \right) \sin \phi = 0$$

For small ϕ

$$\ddot{\phi} + \left(-\frac{g}{l} + \frac{4\omega_0^2 y_0}{l} \cos 2\omega t \right) \phi = 0$$

Compare with Eq. (9) Sec. 1.10. which is

$$\frac{d^2 y}{dz^2} + (a - 2b \cos 2z) y = 0$$

which results in

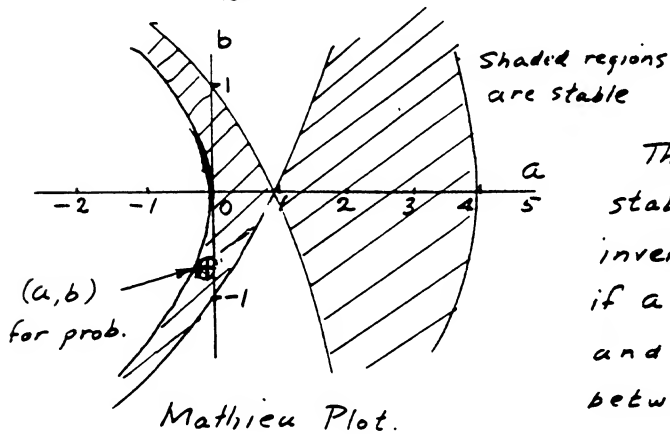
$$z = \omega t, \quad y = \phi, \quad dz^2 = \omega^2 dt^2$$

Rewrite eq.

$$\omega^2 \frac{d^2 \phi}{dz^2} + \left(-\frac{g}{l} + \frac{4\omega_0^2 y_0}{l} \cos 2z \right) \phi = 0$$

$$\frac{d^2 \phi}{dz^2} + \left(-\frac{g}{\omega^2 l} + \frac{4y_0}{l} \cos 2z \right) \phi = 0$$

$$\therefore a = -\frac{g}{\omega^2 l} \quad \text{and} \quad b = -\frac{2y_0}{l}$$



The plot indicates that stable oscillations of the inverted pendulum is possible if a is a small number and b is also negative between 0 & -1, as shown by the point \bullet .

14-40

New length after displ is



$$l = l_0 \left(1 + \frac{x^2}{l_0^2}\right)^{1/2} \approx l_0 \left(1 + \frac{1}{2} \frac{x^2}{l_0^2}\right)$$

Let T_0 = initial tension. Since increase in length is $\frac{x^2}{l_0}$, the increase in tension is $K \frac{x^2}{l_0}$. Total tension = $(T_0 + K \frac{x^2}{l_0})$

Eq. of motion

$$m \ddot{x} = - \frac{x}{l} 2 \left(T_0 + K \frac{x^2}{l_0}\right) \quad \frac{x}{l} \approx \frac{x}{l_0}$$

$$m \ddot{x} + \left(\frac{2T_0}{l_0}\right)x + \left(\frac{2K}{l_0^2}\right)x^3 = 0$$

Assume sol. $x = x_1 + \mu x_2 + \dots$ where μ = arbitrary parameter and $x_2 \ll x_1$. Then $x^3 \approx x_1^3 + 3\mu x_1^2 x_2$. Subst. in D.E.

$$m \ddot{x}_1 + \frac{2T_0}{l_0} x_1 + \alpha x_1^3 = 0 \quad \alpha = \frac{2K}{l_0^2}$$

$$\mu \left[m \ddot{x}_2 + \frac{2T_0}{l_0} x_2 + 3\alpha x_1^2 x_2 \right] = 0$$

If α is small, then $x_1 = A \cos \omega_n t$, $\omega_n = \sqrt{\frac{2T_0}{l_0}}$ and

2nd eq. becomes

$$m \ddot{x}_2 + \frac{2T_0}{l_0} x_2 + (3\alpha A^2 \cos^2 \omega_n t) x_2 = 0$$

$$m \ddot{x}_2 + \left[\left(\frac{2T_0}{l_0} + \frac{3}{2} \alpha A^2\right) + \frac{3}{2} \alpha A^2 \cos 2\omega_n t \right] x_2 = 0$$

which is Mathieu eq.

14-41

u.

```
»t0=0
```

```
t0 =
```

```
0
```

```
»tf=10
```

```
tf =
```

```
10
```

```
»x0=[0, pi/3]
```

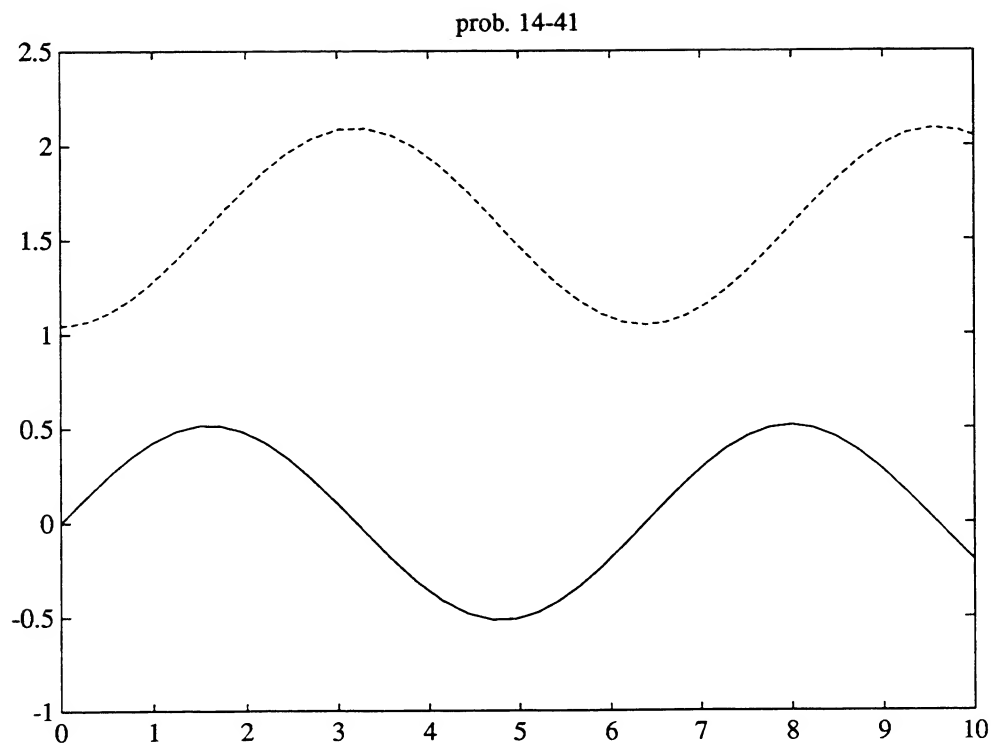
```
x0 =
```

```
0    1.0472
```

```
»[t,x]=ode23('probl441',t0,tf,x0);
```

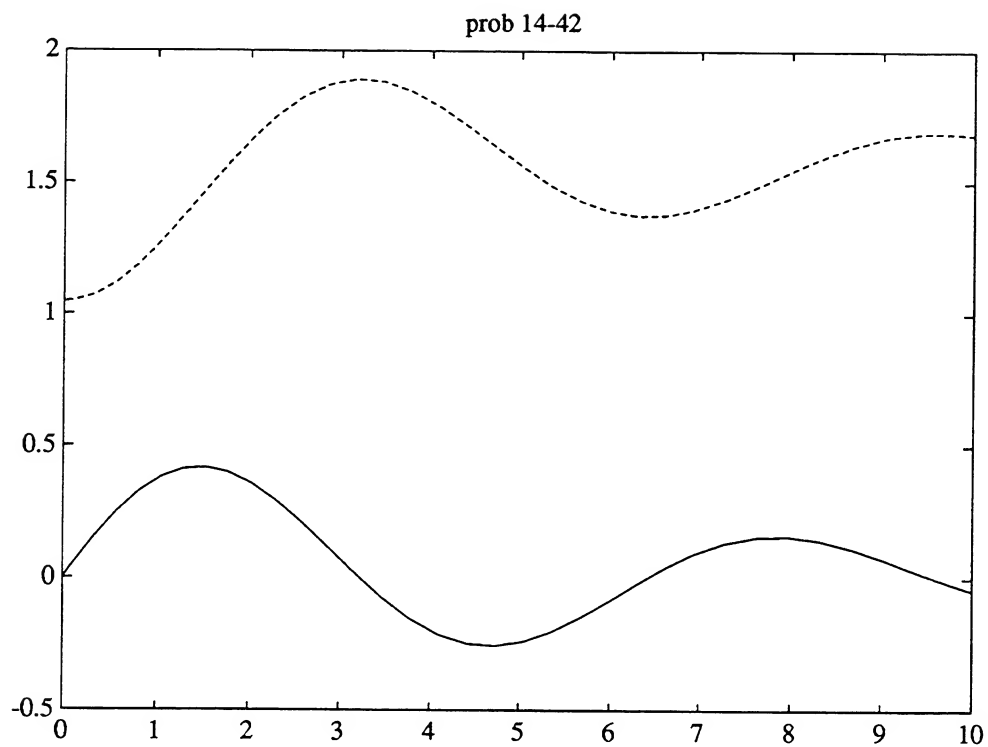
```
»plot(t,x)
```

```
function xdot=probl441(t,x)
xdot=[sin(x(2));x(1)];
```



14-42
Input the initial condition as in problem 14-41

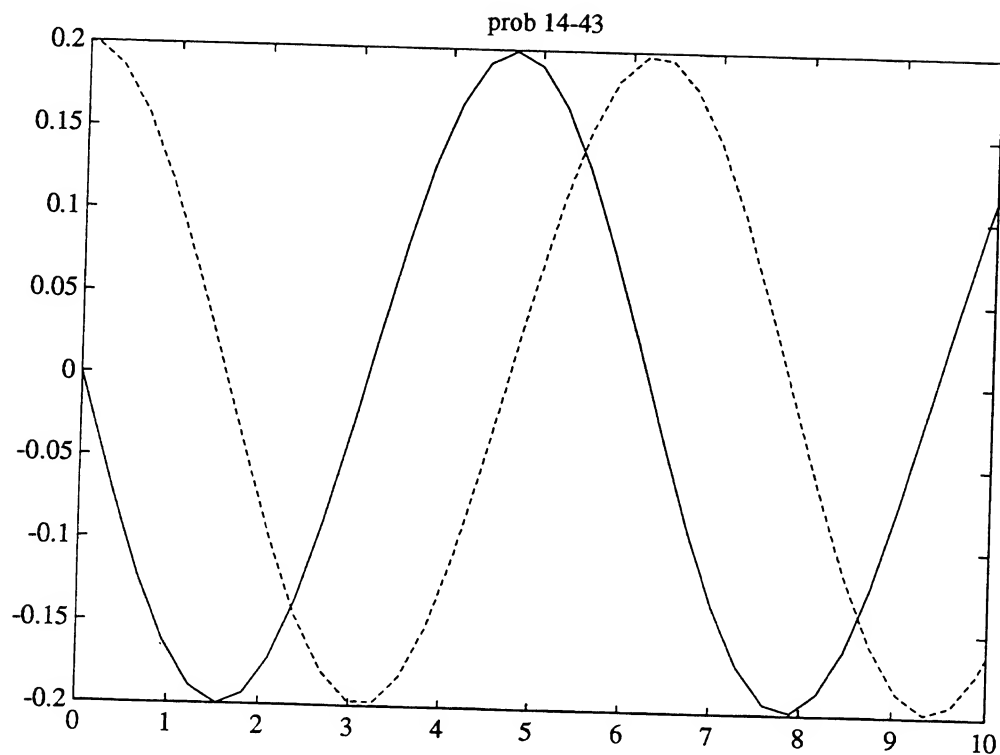
```
function xdot=probl442(t,x)
xdot=[sin(x(2))-.3*x(1);x(1)];
```



14-43

Input The initial conditions as in problem 14-4,

```
function xdot=probl443(t,x)
xdot=[-x(2)-.2*x(2)^3;x(1)];
```



14-44

≡

```
clear
global m g l F0 w
'Problem 14-44 Solution'
m=input('Enter the value of the mass (kg) : ')
l=input('Enter the value of the pendulum length (m) : ')
F0=input('Enter the value of the forcing amplitude (N) : ')
w=input('Enter the value of the forcing frequency (rad/s) : ')
x0=input('Enter the two dimensional initial condition [x xd] : ')
t0=input('Enter the value of the initial time (s) : ')
Tf=input('Enter the value of the final time (s) : ')
g=9.81;
xp0=[x0 w*t0];
T0=0;
[T,xp]=ode45('p1444d',T0,Tf,xp0);
m=size(xp);n=m(1,1);
j=1;
y1(1)=xp(1,1);
y2(1)=xp(1,2);
plot(y1(1),y2(1),'.')
hold on
for i=2:n
    xpo(3)=2*pi*j+xp0(3);
    if xp(i,3)-xpo(3)>0 & xp(i-1,3)-xpo(3)<0
        j=j+1;
        s1=(xp(i,1)-xp(i-1,1))/(xp(i,3)-xp(i-1,3));
        y1(j)=s1*(xpo(3)-xp(i-1,3))+xp(i-1,1);
        s2=(xp(i,2)-xp(i-1,2))/(xp(i,3)-xp(i-1,3));
        y2(j)=s2*(xpo(3)-xp(i-1,3))+xp(i-1,2);
        plot(y1(j),y2(j),'.')
    end
    hold on
    if xp(i,3)==xpo(3)
        j=j+1;
        y1(j)=xp(i,1);
        y2(j)=xp(i,2);
        plot(y1(j),y2(j),'.')
    end
    hold on
end
xlabel('theta');ylabel('thetadot');title('Problem 14-44:Poincare Section')
```

14-45

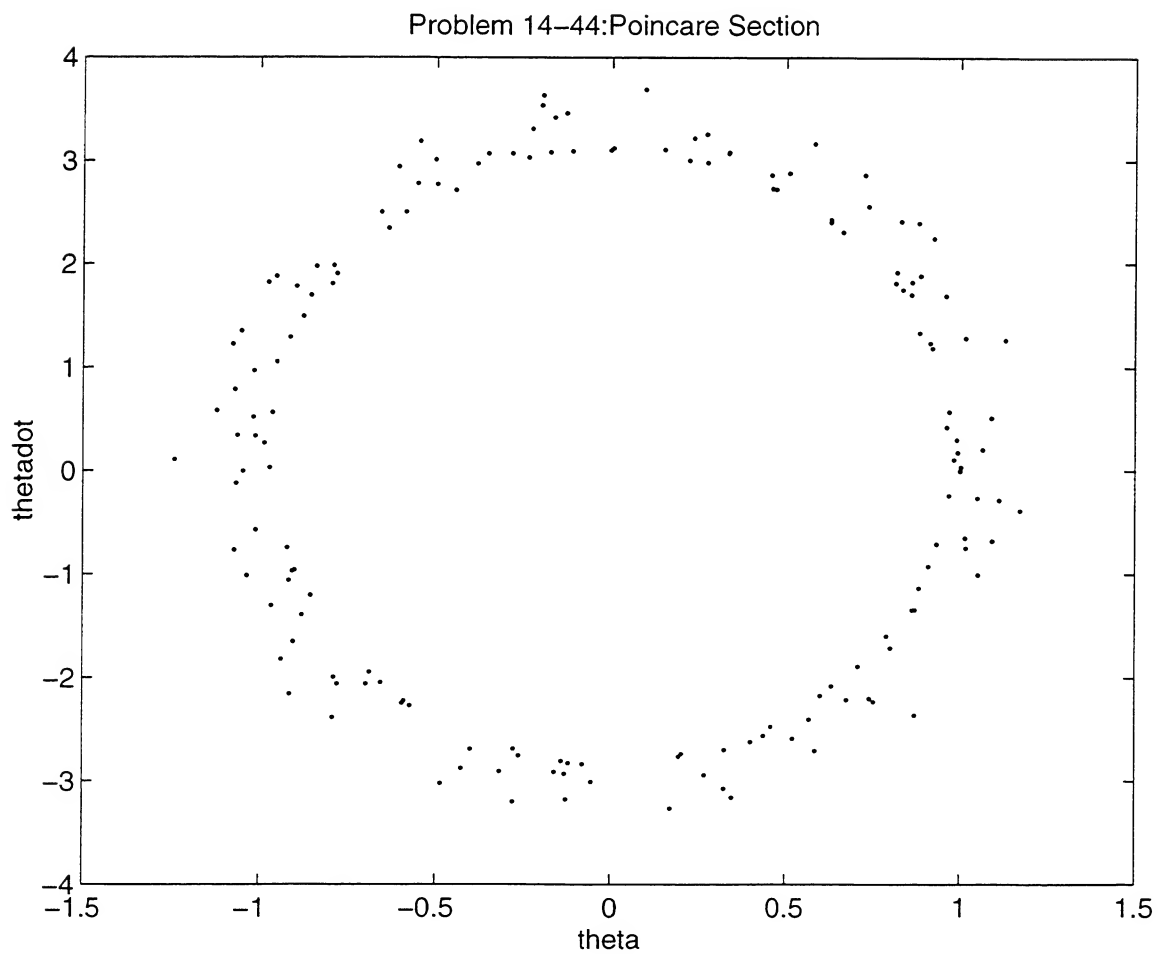
=

```
clear
global m c k mu F w
'Problem 14-45 Solution'
m=input('Enter the value of the mass (kg) : ')
c=input('Enter the value of the damping coefficient (N.m/s) : ')
k=input('Enter the value of the spring stiffness (N/m): ')
mu=input('Enter the value of mu (N/m^3) with its sign : ')
F=input('Enter the value of the forcing amplitude (N) : ')
w=input('Enter the value of the forcing frequency (rad/s) : ')
x0=input('Enter the two dimensional initial condition [x xd] : ')
t0=input('Enter the value of the initial time (s) : ')
Tf=input('Enter the value of the final time (s) : ')
xp0=[x0 w*t0];
T0=0;
[T,xp]=ode45('p1445d',T0,Tf,xp0);
m=size(xp);n=m(1,1);
j=1;
y1(1)=xp(1,1);
y2(1)=xp(1,2);
plot(y1(1),y2(1),'.')
hold on
for i=2:n
    xpo(3)=2*pi*j+xp0(3);
    if xp(i,3)-xpo(3)>0 & xp(i-1,3)-xpo(3)<0
        j=j+1;
        s1=(xp(i,1)-xp(i-1,1))/(xp(i,3)-xp(i-1,3));
        y1(j)=s1*(xpo(3)-xp(i-1,3))+xp(i-1,1);
        s2=(xp(i,2)-xp(i-1,2))/(xp(i,3)-xp(i-1,3));
        y2(j)=s2*(xpo(3)-xp(i-1,3))+xp(i-1,2);
        plot(y1(j),y2(j),'.')
    end
    hold on
    if xp(i,3)==xpo(3)
        j=j+1;
        y1(j)=xp(i,1);
        y2(j)=xp(i,2);
        plot(y1(j),y2(j),'.')
    end
    hold on
end
end
xlabel('x');ylabel('xdot');title('Problem 14-45:Poincare Section')
```

14-45 cont

```
function xdot=p1444d(t,x)
global m g l F0 w
xdot=[x(2);
      -(g/l)*sin(x(1))+(F0/(m*l))*sin(x(3));
      w];
```

$$m=1=F0=w=1, \quad x_0=[1 \quad 0], \quad t_0=0, \quad T_f=1000$$



14-45 cont

ε

```
function xdot=p1445d(t,x)
global m c k mu F w
xdot=[x(2);
      (1/m)*(-c*x(2)-k*x(1)-mu*x(1)^3+F*cos(x(3)));
      w];
```

$$x_0 = [0 \ 0], \ t_0 = 0, \ T_f = 1000$$

